Contagion in Debt and Collateral Markets *

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Abstract

This paper investigates contagion in financial networks through collateralized debt and its effects on social welfare. Our model incorporates contagion through both counterparty debt exposures and endogenous collateral asset pricing. We find that collateral mitigates counterparty exposures and reduces social inefficiency when faced with negative shocks, but not always. We also show the importance of the interaction between the level of collateral and network structure as contagion can change dramatically depending on that interaction. The model also provides policy-relevant collateral-to-debt ratios (haircuts) to attain robust and fully insulated macroprudential states for any network and also the optimal collateral ratio to attain full insulation for a specific network.

Keywords: collateral, financial network, fire sale, systemic risk

JEL Classification Numbers: D49, D53, G01, G21, G33

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1. Introduction

This paper studies the social welfare implications of negative shocks to exogenous financial networks in collateralized debt markets. The Global Financial Crisis (GFC) was exacerbated by a failure of collateralized debt markets such as the repurchase agreement (repo) markets (Gorton and Metrick, 2012; Covitz et al., 2013). Furthermore, the volume of collateralized debt markets is very large—the average daily amount of cash and collateral traded in bilateral repo markets in the U.S. is more than $4 trillion, and the global market for securities lending had more than $3 trillion in outstanding contracts in 2021.[1] Hence, properly characterizing the contagion in such markets is important for both academics and policymakers.

Typical collateralized debt takes the form of a one-to-one relationship between a borrower and a lender. If the value of the collateral is greater than the face value of the debt, then the payment is always made in full. However, if the value of the collateral is less than the face value of the debt, then the payment depends on both the price of the collateral and the cash balance of the borrowing counterparty. Therefore, a collateralized debt network has two transmission channels of shocks, the collateral price channel and the counterparty channel, and the interaction of network structure and collateral prices can dramatically alter a network's systemic risk and, thus, welfare.[2]

We develop a network model with collateral featuring these two channels of propagation. Typical financial network models in the literature investigate only contagion of liquidity shortages through the counterparty channel or price-mediated losses of common asset holdings. However, payments from collateralized debt contracts depend on both channels because the collateral price changes endogenously and simultaneously, which is accounted for in our framework.

[1] https://www.sifma.org/wp-content/uploads/2022/02/SIFMA-Research-US-Repo-Markets-Chart-Book-2022.pdf, https://home.treasury.gov/system/files/261/FSOC2021AnnualReport.pdf

[2] For example, the collapse in prices of subprime mortgages during the GFC was exacerbated by the bankruptcy of Lehman Brothers, which spread the initial losses to Lehman's counterparties and further decreased asset prices (Singh, 2017).

The model is based on an economy of n agents, who are endowed with an asset that can be traded and used as collateral in a competitive market. The price of the asset is endogenously determined in a competitive Walrasian market, and it is a function of the total remaining wealth in the economy. The payoff of the collateralizable asset is common knowledge and realized in the final period. In the first period, agents borrow from each other using bilateral one-period debt contracts, specifying the amount of debt and collateral. The structure of these liabilities is described by a collateralized debt network, which is exogenously given. Agents also invest in a long-term project that generates a non-pledgeable return at the final period. Liquidating the long-term project is costly and thus socially inefficient. However, if agents receive a negative liquidity shock, they may have to liquidate their long-term projects to pay their debt. If an agent's net wealth is still negative after liquidations, then the agent defaults, which can trigger additional liquidations through the network.

This model is consistent with the literature, as it extends and reproduces the results of Acemoglu et al. (2015), who model unsecured debt networks. Acemoglu et al. (2015) show that there exists a phase transition, "robust yet fragile," property of contagion. If the liquidity shocks are small, having more links can improve social surplus because more lending relationships can diversify losses. If the shocks are large, each link becomes a source for contagion, so having more links can decrease social surplus.

The coexistence of collateral and bilateral debt relationships in our model generates novel insights, as we can analyze the interaction across the amount of collateral, the network structure, and the size of the shock. If collateral is enough, any network structure is insulated from contagion.[3] If collateral is not enough, there will be an interaction between the amount of collateral and network structure, leading to extremely different levels of inefficient liquidations and social surplus. The collateral price could also fall below its fundamental value, because the counterparty losses, due to insufficient collateral, cause additional agents to default and put downward pressures on the collateral price. The overall economy will suffer

[3] This result is in line with real-world markets; for example, repo collateral is exempt from the automatic stay of bankruptcy provisions and prevents further spillovers.

from inefficient liquidations due to defaults, amplified by contagion. In such a case, only a network with limited interconnectedness across agents can prevent a full collapse.

Our results highlight the fragility of collateral's role in reducing counterparty exposures. When defaults occur, collateral can act as a buffer as long as the counterparty exposures are limited. However, if the counterparty exposures are large relative to collateral, then the collateral price can plunge to zero, leaving all the contracts unsecured. Therefore, the equilibrium collateral price shows a phase transition (bang-bang) property.

Finally, the model provides insights to policymakers regarding collateral requirements. We obtain two thresholds of the collateral-to-debt ratio (collateral ratio), applicable to any network structure, required to prevent either any contagion at all or contagion through collateral price, respectively. Furthermore, we find the optimal (minimally required) collateral ratio that can prevent contagion for a given network structure. Thus, we provide three different policy options for policymakers to choose, based on the tradeoff between financial stability (by enforcing more collateral) and collateral efficiency as well as information on the network structure.

The first contribution of this paper is developing a model with both full-recourse debt contract and collateral channels of contagion, which is the first attempt in the literature. The literature on contagion in financial networks through payments, pioneered by Allen and Gale (2000) and Eisenberg and Noe (2001), and extended by Acemoglu et al. (2015), usually focuses on the tradeoff between diversification and the contagion channel from having more links. This paper suggests that the tradeoff can change depending on the collateral-to-debt ratio and the aggregate shocks since the contracts are collateralized and the collateral asset price is endogenous. The key difference is that we account for the excess cash inflows of non-defaulting agents to track the changes in the collateral price that affect defaults and liquidations through the collateralized debt contract structure.

The feedback from agents' net wealth to the collateral price is crucial in this paper. Other papers consider the interaction between counterparty and price channels, such as Rochet and

Tirole (1996); Cifuentes et al. (2005); Gai et al. (2011); Capponi and Larsson (2015); and Di Maggio and Tahbaz-Salehi (2015). In contrast, we incorporate explicit collateral and an endogenous price channel of contagion for the underlying collateral.

The endogenous price determination in this paper is based on the literature on general equilibrium with collateralized debt, as in Geanakoplos (1997), Geanakoplos (2010), and Fostel and Geanakoplos (2015). This paper contributes to this literature by adding network contagion and analyzing the interaction of counterparty risks and prices.

Many financial network models have an equilibrium with overlapping asset and counterparty portfolios (Cabrales et al., 2017). The literature documents fire sales in financial markets, which implies that sales of a financial institution can depress asset prices and lead to more sales from others, further depressing both the market price and balance sheets (Chen et al., 2010; Jotikasthira et al., 2012; Greenwood et al., 2015; Duarte and Eisenbach, 2021). Our model extends the fire-sales model and incorporates complicated non-linear effects of fire sales by analyzing how asset prices can affect counterparty payments and vice versa.

This paper generalizes the non-recourse contracts in Chang (2021) to full recourse contracts to analyze more general and realistic borrower default contagion while abstracting out from network formation in Chang (2021). This paper differs from Ghamami et al. (2022), who study how different contract termination rules affect the spread of losses and defaults in financial networks with collateral, by analyzing how network structures interact with collateral requirements.

Finally, this paper is also related to the literature on the role of collateral. Only the aggregate level of collateral matters in the general equilibrium literature because contracts are fully anonymized and diversified. Demarzo (2019) addresses that collateral can be a cost-efficient commitment device, and Donaldson et al. (2020) argue that secured debt prevents debt dilution. This paper shows how explicitly designated individual collateral, which is consistent with the practices in the real world, matters when counterparty risk is involved through the roles of mitigating and amplifying channels of counterparty contagion.

2. Model

2.1. Agents and Goods

There are three periods $t = 0, 1, 2$ and two goods, cash and an asset, denoted as e and h, respectively. Cash is the only consumption good, the numeraire good, and storable. The asset can be used as collateral at $t = 0$ and yields s amount of cash at $t = 2$. Agents gain no utility from just holding the asset. All agents know the value of the asset payoff s at $t = 1$; however, the asset payoff is realized at $t = 2$.

There are n different agents, and the set of all agents is $N = \{1, 2, \ldots, n\}$. Agents are risk neutral, and their utility is determined by how much cash they consume at $t = 2$. Each agent is investing in a long-term investment project that will give ξ amount of cash at $t = 2$ if it is held until maturity. The payoff from this long-term project is not pledgeable. Agent j can partially liquidate the project by $l_j \in [0, \xi]$ amount at $t = 1$ to receive the scrap value of ζl_j in terms of cash, where $0 \leq \zeta < 1$ represents the liquidation efficiency.

All information is common knowledge, and the markets for both goods are competitive Walrasian markets. Thus, agents are price-takers, and there is no asymmetric information. The price of the asset is p_t for $t = 0, 1, 2$. From now on, we use p instead of p_1 for the price of the asset at $t = 1$, as our main focus is analyzing the contagion in $t = 1$.

2.2. Collateralized Debt Network

At $t = 0$, each agent $j \in N$ holds e_j amount of cash and h_j amount of assets, which are exogenously given, until $t = 1$. At $t = 1$, agents can buy or sell the asset in a competitive market. Also at $t = 0$, agents borrow or lend cash using assets as collateral. All borrowing contracts are a one-period contract between $t = 0$ and $t = 1$ and are exogenously determined. A borrowing contract consists of the promised cash payment, the ratio of collateral posted per one unit of promised cash, and the identities of the borrower and the lender. Denote d_{ij}

as the promised cash amount to pay at $t = 1$ to lender i by borrower j. Denote c_{ij} as the collateral ratio per one unit of promised cash. If borrower j pays back the full amount of promised d_{ij}, then the lender returns the collateral in the amount of $c_{ij}d_{ij}$. Otherwise, the lender either keeps or liquidates the collateral, and the cash value of the collateral is $c_{ij}d_{ij}p$. The lender has to return any excessive value of the collateral to the borrower, $c_{ij}d_{ij}p - d_{ij}$, if there is any. Normalize $c_{ii} = d_{ii} = 0$ for all $i \in N$ without loss of generality.

Define $C = [c_{ij}]$ and $D = [d_{ij}]$ as the matrices of collateral ratios and promised debt payment amounts, respectively. A collateralized debt network is a weighted, directed multiplex graph that is formed by the set of vertices N and links with two layers C and D. A (collateralized debt) network can be summarized by a double (C, D) given at $t = 0$. Denote the total inter-agent liabilities of agent j as $d_j \equiv \sum_{i \in N} d_{ij}$.

We assume that *collateral constraints* and *resource constraints* hold, which implies

$$\sum_{k \in N} c_{jk}d_{jk} + h_j \geq \sum_{i \in N} c_{ij}d_{ij} \qquad \forall j \in N, \tag{1}$$

$$\sum_{i \in N} h_i \geq \sum_{i \in N} c_{ij}d_{ij} \qquad \forall j \in N. \tag{2}$$

The collateral constraint, (1), means that the total amount of collateral a borrowing agent j posts cannot exceed the amount of assets agent j has—either from other agents' collateral that agent j has received as a lender or from the amount of assets agent j purchased outright. This collateral constraint allows reuse (rehypothecation) of collateral.[4] The resource constraint, (2), means that the total amount of assets an agent is posting cannot exceed the total amount of assets in the economy.[5]

Each agent can be hit by a negative liquidity shock in cash at the absolute value of $\epsilon > 0$ at $t = 1$. Agents should pay the liquidity shock first before paying other agents. We interpret

[4]The same collateral can be reused an arbitrary number of times, generalizing typical models of reuse of collateral (Gottardi et al., 2019; Infante, 2019; Park and Kahn, 2019; Infante and Vardoulakis, 2021).

[5]If a resource constraint is not present, then there can be a spurious cycle of collateral justifying any arbitrary amount of collateral circulating in the economy.

ϵ as a senior debt payment to external creditors, who also have linear utility.[6] A realized state of the liquidity shocks is $\omega \equiv (\omega_1, \omega_2, \ldots, \omega_n)$, and the set of all possible states is Ω. For example, $\omega_j = 1$ if agent j is under a liquidity shock, and $\omega_j = 0$ otherwise.

Liabilities other than the liquidity shock are all equal in seniority. Hence, any net wealth left after paying the liquidity shocks will be distributed across all agents on a pro rata basis.

2.3. Timeline and Social Surplus

The timeline of the model, depicted in Figure 1, is the following. Agents' cash and asset holdings as well as debt network are exogenously given at $t = 0$. At the beginning of $t = 1$, asset payoff s is publicly revealed, and liquidity shocks ϵ are realized for some agents. Agents may liquidate their long-term projects if they are short on cash. Agents pay their debt, and collateral is returned to the borrower, if not defaulted. If an agent defaults, any remaining assets in the agent's balance sheet will be distributed to all other creditors on a pro rata basis. At the end of $t = 1$, all agents' final asset holdings are determined. At $t = 2$, the payoff of the asset is realized, and agents consume all the cash they have and gain utility.

Define the utilitarian social surplus as the sum of the payoffs of all agents at $t = 2$,

$$U = \sum_{i \in N} (\pi_i + T_i),$$

where $T_i \leq \epsilon$ is the transfer from agent i to its senior creditors (liquidity shock), which simply transfers to $t = 2$, and π_i is the agent's long-term profit evaluated at $t = 2$, which is consisted of cash payoffs through cash holdings and net debt payments, asset payoffs from asset holdings and net collateral flows, and the remaining amount of the long-term project.[7]

[6] Alternative interpretations of negative liquidity shocks include lower-than-expected short-term returns, a sudden increase in deposit withdrawals, wage expenses, taxes, and fines.

[7] This definition of social surplus is consistent with that of Acemoglu et al. (2015).

Lemma 1. *For any full equilibrium, the social surplus in the economy is equal to*

$$U = \sum_{i \in N} (e_i + h_i s + \xi) - (1 - \zeta) l_i.$$

All proofs are relegated to the appendix. Lemma 1 clarifies that the source of social inefficiency comes from the early liquidation of the long-term project, caused by either insufficient liquidity or a low asset price that makes an asset purchase more profitable than the long-term project.

3. Full Equilibrium

In this section, we define the equilibrium concept and its relevant elements.

3.1. Liquidation and Payment Rules

Let $x_{ij}(p)$ denote the actual payment net of collateral to agent i from agent j when the asset price is p at $t = 1$. This payment will be defined later in (6). The argument p is often omitted from now on. The total *cash inflow* of agent j before liquidating the project is

$$a_j(p) \equiv e_j + h_j p + \sum_{k \in N} c_{jk} d_{jk} p - \sum_{i \in N} c_{ij} d_{ij} p + \sum_{k \in N} x_{jk}(p), \tag{3}$$

where the second term is the market value of j's direct asset holdings; the third and fourth terms are the market values of collateral assets posted by j's borrowers and posted to j's lenders, respectively; and the fifth term is the actual payment net of collateral from j's borrowers. The total amount of liabilities net of collateral posted for agent j is

$$b_j(p, \omega) \equiv \sum_{i \in N} (d_{ij} - c_{ij} d_{ij} p) + \omega_j \epsilon, \tag{4}$$

which can be considered the required total *cash outflow*. Note that the first term of the right-hand side can be negative if the contract is overcollateralized—that is, $d_{ij} < c_{ij}d_{ij}p$. The function argument ω is often omitted for simplicity from now on.

If $a_j(p) \geq b_j(p)$, then $x_{ij} = d_{ij} - c_{ij}d_{ij}p$ for any $i \neq j$. If $a_j(p) < b_j(p)$, then agent j liquidates the long-term project to meet liabilities to others. Moreover, if the price of the asset is very low, the return from purchasing the underpriced asset, s/p, can be greater than the long-term return of the project, $1/\zeta$, and all agents will liquidate their projects regardless of their obligations. Note that agent j is indifferent between liquidating more projects to purchase more assets and keeping the projects once $p = s\zeta$.

Mathematically, agent j's liquidation decision, $l_j(p) \in [0, \xi]$, is

$$l_j(p) = \begin{cases} \underline{l_j}(p) \equiv \left[\min\left\{\frac{1}{\zeta}\left(b_j(p) - a_j(p)\right), \xi\right\}\right]^+ & \text{if} \quad p > s\zeta \\ \ell \in [\underline{l_j}(p), \xi] & \text{if} \quad p = s\zeta \\ \xi & \text{if} \quad p < s\zeta, \end{cases} \tag{5}$$

where $[\cdot]^+ \equiv \max\{\cdot, 0\}$, which guarantees that an agent cannot liquidate the long-term project for a negative amount if $a_j(p) > b_j(p)$. The liquidation decision follows the *liquidation rule* if equation (5) holds. See Appendix B for the full description of the agent's optimization problem and derivation of its solution, which results in the liquidation rule.

Given the liquidation rule, the actual payment to lender i from borrower j can be determined. If agent j can pay all of the obligations (possibly by liquidating the project), then j can pay the promised amount such that $x_{ij}(p) = d_{ij} - c_{ij}d_{ij}p$ as lender i returns the collateral to borrower j, as depicted in the top right of Figure 2. If the total value of collateral $c_{ij}d_{ij}p$ is greater than the debt amount d_{ij}, then the actual net payment $x_{ij}(p)$ can be negative because the more valuable collateral is returned to the borrower from the lender's balance, as depicted in the bottom right of Figure 2. Agent j *defaults* if the payment net of collateral is less than the promised payment, $x_{ij}(p) < d_{ij} - c_{ij}d_{ij}p$ for some $i \in N$. The extreme case is agent j being unable to pay the liquidity shock even after full liquidation in which the actual payment will be $x_{ij}(p) = 0$, and the lender keeps the collateral, as depicted in the middle-right section of Figure 2. In an intermediate case, agent j can pay the liquidity shock but cannot pay the inter-agent debt in full. Under such a case, j's remaining wealth is paid out on a pro rata basis. This interaction is formulated as the following *payment rule*:

$$x_{ij}(p) = \min\left\{d_{ij} - c_{ij}d_{ij}p\ ,\quad q_{ij}(p)\left[a_j(p) + \zeta l_j + \sum_{i\in N}[c_{ij}d_{ij}p - d_{ij}]^+ - \omega_j\epsilon\right]^+\right\}, \tag{6}$$

where $q_{ij}(p)$ is a weight under the *weighting rule*

$$q_{ij}(p) = \frac{[d_{ij} - c_{ij}d_{ij}p]^+}{\sum_{k\in N}[d_{kj} - c_{kj}d_{kj}p]^+} \tag{7}$$

for the pro rata basis. Note that if weights are not defined ($\sum_{k\in N}[d_{kj} - c_{kj}d_{kj}p]^+ = 0$), the weighting rule is never used, because any lender will be paid in full.

3.2. Fire Sales and Market Clearing

For a given network and state realization, or economy, $(N, C, D, e, h, s, \omega)$, where $e \equiv (e_1, e_2, \ldots, e_n)$ and $h \equiv (h_1, h_2, \ldots, h_n)$, the *net wealth* of agent j is

$$\begin{aligned} m_j(p) &\equiv a_j(p) + \zeta l_j(p) - b_j(p) \\ &= e_j + h_jp + \sum_{k\in N} c_{jk}d_{jk}p + \zeta l_j(p) - \omega_j\epsilon - \sum_{i\in N} d_{ij} + \sum_{k\in N} x_{jk}(p) \end{aligned} \tag{8}$$

under the liquidation and payment rules. Equation (8) consists of the following: cash holdings, the market value of the asset holdings, the market value of collateral received, cash from liquidating the long-term project, a negative liquidity shock, the total payment to be paid, and the actual net payment received. If $m_j(p) < 0$, then agent j defaults.

Denote the *fire-sale* amount of agent j as

$$\phi_j(p) = \min\left\{[h_j p - m_j(p)]^+, h_j p\right\}. \tag{9}$$

If agent j's net wealth subtracted by j's asset holdings, $m_j(p) - h_j p$, is enough to cover all of the payments (positive), then $\phi_j(p) = 0$ (no fire sales). If agent j's net cash flow is not enough without the sale of asset holdings, then $\phi_j(p) > 0$. If the cash shortage exceeds the total asset holdings ($h_j p - m_j(p) > h_j p$), then the fire-sale amount reaches its upper bound $\phi_j(p) = h_j p$. Note that default occurs when an agent does not have sufficient liquidity and has to liquidate all the asset holdings, implying $\phi_j(p) = h_j p$.

The market for the asset is a perfectly competitive Walrasian market. Unless there is not enough cash to purchase all of the asset sales in the market at the asset's fundamental value s, the market price will always be the fair value s. However, if there is not enough cash in the market, then the asset price can go below its fundamental value as $p < s$, which is a *liquidity-constrained price*. Under such a case, the market clearing condition becomes a cash-in-the-market pricing condition. The *market clearing condition* can be summarized as

$$\begin{aligned} \sum_{j \notin \mathcal{D}(p)} [m_j(p) - h_j p]^+ &= \sum_{i \in N} \phi_i(p) \qquad \text{if } \ 0 \le p < s \\ \sum_{j \notin \mathcal{D}(p)} [m_j(s) - h_j s]^+ &\ge \sum_{i \in N} \phi_i(s) \qquad \text{iff } \quad p = s, \end{aligned} \tag{10}$$

where $\mathcal{D}(p)$ is the set of agents who default under price p.

For the given rules, the definition of the equilibrium is as follows.

Definition 1. *For given $(N, C, D, e, h, s, \omega)$, if liquidation decisions $\{l_j(p)\}$ satisfy the liquidation rule (5), payments $\{x_{ij}(p)\}$ satisfy the payment rule (6), $\{m_j(p)\}$ is determined by the net wealth equation (8), the fire-sale amount $\{\phi_j(p)\}$ is determined by equation (9), and price p clears the market as in (10), then $(\{x_{ij}\}, \{l_j\}, \{m_j\}, \{\phi_j\}, p)$ is a full equilibrium.*

The notion of this full equilibrium is a generalization of the payment equilibrium in

Acemoglu et al. (2015), which is based on Eisenberg and Noe (2001). In contrast to these papers, agents in our model not only have financial liabilities and liquidation of projects, but they also have posted collateral, and the price of the collateral asset is determined endogenously. Therefore, both the debt and collateral markets have spillovers to each other.

The following proposition shows that the full equilibrium always exists.

Proposition 1. *For any given economy $(N, C, D, e, h, s, \omega)$, a full equilibrium always exists and is generically unique for a given equilibrium price.*[8] *Furthermore, there exists a full equilibrium with the highest price among the set of full equilibria.*

Even though there could be multiple equilibria, each equilibrium price has a (generically) unique full equilibrium, and there exists a maximum full equilibrium that has the highest market clearing price among the set of equilibria. From now on, we focus on the results of the maximum full equilibrium, as in Elliott et al. (2014).

3.3. Discussion

The model incorporates the role of explicit collateral, as agents can settle the payments by giving up their collateral to their lenders. This is in line with the standard repo contracts such as the Securities Industry and Financial Markets Association's (SIFMA) Master Repurchase Agreement (MRA), used by most U.S. dealers, and the SIFMA/International Capital Market Association (ICMA) Global Master Repurchase Agreement (GMRA), used for non-U.S. repos (Baklanova et al., 2015). According to both the SIFMA MRA and SIFMA/ICMA GMRA, after determining the market value of the collateral, all repo exposures between the two counterparties are netted off, and whoever owns the residual sum must pay it by the next business day, including the interest on late payment.[9] Hence, the lender has recourse to the borrower's balance sheet and can claim any payment due net of the market value of the

[8] In other words, there are multiple equilibria only for a non-generic set of parameters, n, e, ω, ζ, ξ, and ϵ, which are all on a line over a multidimensional Euclidean space.

[9] See https://www.sifma.org/wp-content/uploads/2017/08/MRA_Agreement.pdf and https://www.sifma.org/wp-content/uploads/2017/08/Global-Master-Repurchase-Agreement.pdf.

collateral (Gottardi et al., 2019). The non-defaulting party may either immediately sell in a recognized market at prices the non-defaulting party reasonably deems satisfactory or give the defaulting party credit for collateral in an amount equal to the price obtained from a generally recognized source.[10] The non-defaulting party may choose the latter option when the market is under stress, as additional sales of collateral would only decrease the price and the value of the collateral is greater than the current market price.

Likewise, a debt obligation between a lender and borrower in our model references the market price of the collateral, not the fair value s, regardless of whether the lender sells off the asset. While we do this to reflect the default and settlement procedures in the real world, lenders would not accept the collateral at par to begin with, as they would rather demand the full debt amount in cash to purchase potentially cheaper assets priced at market value at $t = 1$.

The property of collateral directly covering the debt payment is crucial for our results. For example, if netting the debt with collateral was not possible, then after a negative liquidity shock to the system, all assets posted as collateral would be put on fire sale as all agents liquidate their collateral simultaneously to raise cash for payment obligations. This version of the model is equivalent to having no collateral at all, as in Acemoglu et al. (2015), because collateral plays no direct role in shaping debt. Indeed, this is not the case in the real world, as market participants typically designate particular collateral and can effectively pay their obligations by giving up their collateral as previously described. In other words, collateral plays the role of money across the debt network when agents pay their liabilities.

4. Contagion, System Risk, and Social Surplus

In this section, we study how the interaction between the collateral ratio and the network structure determines the extent of contagion, systemic risk, and social surplus.

We focus on *regular* networks in which the total inter-agent claims and liabilities of

[10]The Master Securities Loan Agreement also states similar procedures (Baklanova et al., 2015).

all agents are equal—that is, $\sum_{i \in N} d_{ij} = \sum_{i \in N} d_{ji} = d$ for all $j \in N$ for some $d \in \mathbb{R}^+$. Also, assume that all agents hold the same amounts of cash and assets as $e_i = e_0 > 0$ and $h_i = h_0 > 0$ for all $i \in N$. Similarly, we assume that all agents have the same uniform collateral ratio, $c_{ij} = c$ for all $i, j \in N$. This homogeneity assumption guarantees that any variation in systemic risk is due to the level of collateral and the interconnectedness of agents while abstracting away from effects from size, balance sheet, or hierarchical heterogeneity. For simplicity, we follow the benchmark case in Acemoglu et al. (2015) and assume $\zeta = 0$, which is the limit case of $\zeta \to 0$. Almost all of our results can be generalized to a non-trivial liquidation amount, $\zeta > 0$, as we show in Section 5.2. Finally, we assume that only one agent receives the liquidity shock, so $\omega_j = 1$ for some agent j and $\omega_i = 0$ for all $i \neq j$, and the size of the shock is $\epsilon \in (e_0 + h_0 s + \zeta\xi, \infty)$. The lower bound guarantees that, absent any payments from other agents, a shocked agent is unable to pay the liquidity shock even with fire sales of collateral at the best price. This simple setup allows us to examine the financial contagion over both collateral and debt markets for each network in the most intuitive way. We will discuss relaxing these assumptions in Section 5.

Before we describe the results, we define a few important concepts and networks.

Definition 2. *For a fixed (N, e, h, s, Ω), consider two networks (C, D) and $(\tilde{C}, \tilde{D})$.*

1. *(C, D) is more stable than $(\tilde{C}, \tilde{D})$ if $EU \geq E\tilde{U}$, where E is the expectation over ω.*

2. *(C, D) is more resilient than $(\tilde{C}, \tilde{D})$ if $\min_{\omega \in \Omega} U \geq \min_{\omega \in \Omega} \tilde{U}$.*

The two notions compare the expected and worst-case social surplus of a given network.

We define the complete network in which every agent owes the same amount to each other, $d/(n-1)$, and has the highest number of links and the ring network in which every agent borrows all their debt from one other agent.[11] For example, agent 1 has to pay d to agent 2, who has to pay d to agent 3, and so forth. The ring network has the lowest number of links for a connected regular network. Figure 3 illustrates the complete and ring networks.

[11]The ring network is the same as the circle network analyzed in Caballero and Simsek (2013).

Definition 3. *A network* (C, D) *is a* δ*-connected network if there exists* $\mathcal{S} \subset N$ *such that* $\max\{d_{ij}, d_{ji}\} \leq \delta d$ *for any* $i \in \mathcal{S}, j \notin \mathcal{S}$.

Definition 4. *A network* $(C, \tilde{D})$ *is a* γ*-convex combination of two networks* $(C, D), (C, D')$ *if and only if* $\tilde{d}_{ij} = \gamma d_{ij} + (1-\gamma) d'_{ij}$ *for any* $i, j \in N$.

A δ-connected network implies that a network can be separated into two subsets of vertices, with the cross-subset links being relatively small—δ or less. The concept of a γ-convex combination of two networks is equivalent to a convex combination of matrices.

4.1. Example

Suppose there are 5 agents that form a ring network as shown in Figure 3. Each agent is endowed with 1 asset, $h_0 = 1$, 2 units of cash, $e_0 = 2$, and an investment project valued at 1 at $t = 0$. The asset has a fair value of $s = 1$, and its price is determined as in (10). Each agent owes a total debt amount of $d = 10$ to the next agent along the ring and posts a collateral amount of $cd = 2$. If the collateral is priced at its fair value, agents need 8 units of cash to fulfill their debt obligation. Suppose that agent 1 is under a liquidity shock.

When the liquidity shock is small, $\epsilon = 2$, agent 1 uses endowed cash to meet the debt obligation to agent 2 by reusing the payment received from agent 5. No agent liquidates, and the asset price remains $p = s = 1$. If the liquidity shock is slightly larger, $\epsilon = 3$, agent 1 must sell the asset to meet the obligation. The total cash from potential buyers, the 4 other agents, is 8, while the supply is 1 from agent 1. Thus, the asset price is $p = s = 1$, and agent 1 receives 1 unit of cash from the fire sale. Agent 1 pays the liquidity shock and fulfills debt obligations to agent 2. No agent liquidates.

If the liquidity shock is $\epsilon = 10$, agent 1 pays the liquidity shock using agent 5's payment in addition to agent 1's cash holdings and cash from asset sales. Agent 1 liquidates and pays agent 2 the remaining wealth of 1. Agent 2's wealth now consists of 1 asset, 2 units of cash, and agent 1's payment of 1, which, in total, is less than 8, the required payment amount to agent 3. Therefore, agent 2 liquidates, sells the asset holding, and pays agent 3 the remaining wealth of 4. Agent 3 also cannot fulfill the debt obligation to agent 4 and therefore liquidates, sells the asset holding, and pays agent 4 the remaining wealth of 7. Agent 4 uses agent 3's payment of 7 and 1 unit of cash holdings to pay agent 5 in full. Agent 5 meets the debt obligation to agent 1 by reusing agent 4's payment. Since we assume the asset price is 1, we check if that is indeed the case. There are 3 assets on sale, and the total remaining cash is 3. Therefore, the price is sustained at 1, its fair value. Since three agents liquidate, the total loss in social surplus is 3.

If the liquidity shock is $\epsilon = 15$, the chain of events (liquidations, selling assets) will remain the same as in the previous case, with the only difference being each agent's remaining net wealth. Agent 1 transfers 0 to agent 2, who transfers 3 to agent 3, who transfers 6 to agent 4. Using agent 3's payment of 6 and 2 units of cash, agent 4 fulfills the debt obligation to agent 5. Then, agent 5 can also pay agent 1 in full. We check the asset price to see if our initial assumption of $p = s = 1$ holds. There are 3 assets on sale, and the total remaining cash is 2. Therefore, the asset price is $\frac{2}{3}$. Now, the value of collateral is $\frac{4}{3}$, less than 2; thus, each agent's debt obligation is larger, $8 + 2/3$ rather than 8. As a result, agents were at a worse position to meet debt obligations. The price diminishes to 0. All agents liquidate, and the loss in social surplus is 5. The results of this example are summarized in Table 1. Note that while the cases in this example describe a sequence of responses, these responses actually occur simultaneously in equilibrium.

4.2. Three Regimes of Contagion

Our first main result is that there are three different regions of the level of collateral, and the role of interconnectedness changes across these different regions. If the level of collateral is high enough, all the contracts are fully covered by collateral regardless of the network structure and the liquidity shock—that is, the economy is in a "fully insulated" regime. If the collateral ratio is not enough to provide full insulation, then propagation still occurs.

Nevertheless, there is no contagion through the collateral channel when the liquidity shock is small or the collateral ratio is relatively high. We call this a "robust regime" because having more links could make the network more stable and resilient. However, if the liquidity shock is large and the collateral ratio is low, then there is contagion through the collateral price channel. We call this a "fragile regime" because having more links could make the network less stable and resilient. The following proposition, visualized by Figure 4, summarizes our results.

Proposition 2. *(Three Regimes of Contagion)*

Suppose that the size of liquidity shock is ϵ, and the collateral ratio is c.

1. *If $\bar{c}(s,n) \equiv \dfrac{1}{\hat{s}} < \dfrac{1}{s\zeta}$, where $\hat{s} \equiv \min\left\{s, \dfrac{(n-1)e_0}{h_0}\right\}$ and $c \geq \bar{c}(s,n)$, then any network is the most resilient and stable network for any ϵ.*[12]

2. *If either $\epsilon < \epsilon^* \equiv ne_0 + \zeta\xi$ or $c \geq \underline{c}(s,n) \equiv [d-(n-1)e_0 + h_0\hat{s}]/(d\hat{s})$ holds, where $\underline{c}(s,n) \leq \bar{c}(s,n)$, then $p = \hat{s}$ and the following holds:*

   (a) The complete network is the most stable and resilient, while the ring network is the least stable and resilient, and liquidations are decreasing in c.

   (b) The γ-convex combination of the complete and ring networks becomes more stable and resilient as γ increases.

3. *If $c < \underline{c}(s,n)$ and $\epsilon > \epsilon^*$, then there exists d^* such that for any $d > d^*$ the following holds:*

   (a) The complete and ring networks are the least stable and least resilient networks, and the corresponding equilibrium asset price is $p = 0$ for both networks.

   (b) For a small enough δ, a δ-connected network is more resilient and stable than the complete network, and the corresponding equilibrium asset price is $p > 0$.

[12]The condition, $1/\hat{s} < 1/s\zeta$, is needed to prevent price-induced fire sales stemming from the disproportional return of the asset (even when there is no default) compared with that of the long-term project.

One interesting implication of Proposition 2 is that the equilibrium asset price is either 0 or s under the ring and complete networks. Since all agents are connected with each other, a liquidity shock will reach all agents in the network. If the collateral price was high, it would have mitigated the total outflow of cash (to the liquidity shock) from the network. However, the remaining net wealth may decline after the outflow of cash, depressing the price. As the discrepancy between debt and collateral $d - cdp$ increases with the decrease in p, every agent needs additional liquidity. This effect further increases fire sales, and the price decreases even further. The dual loop of contagion between the collateral market and debt payment leads to $p = 0$, and all agents default and liquidate.

Similar to the unsecured debt networks in Acemoglu et al. (2015), a δ-connected network is more resilient and stable than the complete network when the collateral ratio is low, as the separated components limit contagion to each other. Hence, assessing financial stability requires monitoring not only the average level of leverage, but also how counterparty exposures are distributed within the network.

Lastly, recall that we are changing the amount of collateral while fixing agents' endowments and total debt amounts. Hence, our analysis highlights how specifying collateral explicitly could change the contagion drastically, ranging from full insulation to total market collapse. This is because the collateral's role in mitigating counterparty exposures by guaranteeing payments can feed back into the network to support the high price of collateral. Thus, our results highlight the importance of modeling explicit collateral as opposed to modeling collateral implicitly tied to an agent's total asset holdings. Further, our results also highlight the fragility of collateral's role in mitigating contagion. As soon as the counterparty exposures exceed the threshold and collateral is not enough, then the collateral price can plummet to zero, leaving all contracts unsecured. Nevertheless, an increase in collateral will never decrease social surplus regardless of which region the economy is in.[13] The reason is that collateral can only reduce counterparty contagion, not exacerbate it, by guaranteeing

[13]See Section 6 as well as Figures 9 and 10 in the Online Appendix.

the amount of cash equivalent to the price multiplied by the amount of collateral. An increase in c clearly does not decrease social surplus in the worst case scenario (i.e. the lower bound of social surplus) when the market value of collateral is zero, $cdp = 0$. Even if $p > 0$, holding the remaining cash in the system as fixed, an increase in c might decrease the overall price of the asset p, but the total amount covered by the market value of collateral, cdp, would remain the same (or even greater if the collateral price was s). In addition, higher c would always decrease the amount of cash drained out of the network (due to liquidity shock) by guaranteeing the payment of cdp from the shocked agent, holding the remaining cash in the system equal. Therefore, our results show that having higher c is always (weakly) better than having lower c.

4.3. Contagion in General Networks and Harmonic Distance

The results can be extended to contagion in general network structures. We use the concept of harmonic distance introduced by Acemoglu et al. (2015) to generalize our contagion results to any general regular network.

Definition 5. *The harmonic distance from agent i to agent j is*

$$\mu_{ij} = 1 + \sum_{k \neq j} \left(\frac{d_{ik}}{d} \right) \mu_{kj}, \tag{11}$$

with the convention that $\mu_{ii} = 0$ for all i.

As noted in Acemoglu et al. (2015), the harmonic distance from agent i to agent j depends not only on how far each of its immediate borrowers is from j, but also on the intensity of their liabilities to i, by d_{ik}/d. This debt-weighted notion for the unsecured debt network in Acemoglu et al. (2015) is surprisingly useful in our model with secured debt.

Proposition 3. *Suppose that agent j is under a negative liquidity shock of $\epsilon > \epsilon^*$. Then, there exists $\mu^*(p) = (d - cdp)/(e_0 + h_0 p)$, and the following holds:*

1. *If there is a nonempty set $\mathcal{S}$ such that agent $i \in \mathcal{S}$ does not default, then the equilibrium price is either $p = s$ or determined by*

$$\mathbb{1}'G\mu_{sj} = \frac{d - cdp}{e_0 + h_0 p}\mathbb{1}'G\mathbb{1} + \frac{nh_0 p}{e_0 + h_0 p}, \tag{12}$$

where μ_{sj} is the vector of harmonic distances from agents in $\mathcal{S}$ to j, G is a $|\mathcal{S}| \times |\mathcal{S}|$ non-singular M-matrix,[14] and $\mathbb{1}$ is a vector of ones. Furthermore, if $\mu_{ij} < \mu^*(p)$, then agent i defaults.

2. *If all agents default, then the equilibrium price is $p = 0$ and $\mu_{ij} < \mu^*(0)$ for all i.*

3. *If $\mu^*(p) < 1$ for the equilibrium price p, then no other agents default.*

This result implies that the harmonic distance from the agent under shock determines both the price of the asset and the extent of contagion. Thus, Proposition 3 implies that the pairwise harmonic distances represent the fragility of a network, generalizing Proposition 2. For example, the harmonic distance between any pair of agents is minimized in the complete network (Acemoglu et al., 2015). Also, in a δ-connected network, there always exists an agent i such that $\mu_{ij} \geq \mu^*(0)$. Finally, $\mu^*(p)$ becomes small enough if c is large enough; hence, any pairwise harmonic distance is large enough to prevent any contagion.

4.4. Optimal Collateral Ratio

The two thresholds, $\underline{c}(s, n)$ and $\bar{c}(s, n)$, in Proposition 2 are applicable to all networks. In this subsection, we provide a tight collateral threshold that incorporates the network structure using the harmonic distance measure. We derive the optimal collateral ratio, c^*, defined as the minimum collateral ratio required to achieve full insulation for a specific network. The optimal collateral ratio highlights the importance of the interaction between collateral and interconnectedness.

[14] If matrix A can be expressed as $A = sI - B$, $s \geq \rho(B)$, $B \geq 0$, where $\rho(B)$ is the spectral radius of B, then A is an M-matrix (Berman and Plemmons, 1979, p. 133). An M-matrix is non-singular if $s > \rho(B)$.

Proposition 4. *(Optimal Collateral Ratio) When $\epsilon > \epsilon^*$ and $d > d^*$, the minimum collateral ratio required to prevent all n agents, except for the agent under liquidity shock, from liquidating is $c^*(s, D) = \dfrac{d - \underline{\mu}(e_0 + h_0\hat{s})}{d\hat{s}}$, where $\underline{\mu} = \min_{i,j} \mu_{ij}$ such that $i \neq j$.*

The functional form of $c^*(s, D)$ implies that as the minimum harmonic distance within a network increases, less collateral is required to achieve a fully insulated state.

4.5. Policy Implications

Our results show that collateral could be an effective way to manage systemic risk and improve social surplus. However, collateral may fail to play such a role depending on the liquidity shock size and network structure. Therefore, policymakers may mandate a certain level of collateralization or leverage to ensure financial stability.

Our results provide multiple collateral thresholds for policymakers facing varying situations and tradeoffs. The policymaker can reference the thresholds that are applicable to any network structure when there is limited information on the network structure or the structure is highly volatile. If the policymaker prefers full insulation and is less concerned over the allocative efficiency or scarcity of collateral, they could mandate the minimum collateral ratio to be $\bar{c}(s, n)$. However, if the policymaker is more concerned with collateral efficiency and can allow for some debt contagion, they could mandate a minimum collateral ratio closer to $\underline{c}(s, n)$. If the network structure is known and stable, then $c^*(s, D)$ could be mandated, which is the precise threshold in which the entire network is fully insulated without inefficient overuse of collateral.

5. Extensions

5.1. Aggregate Shocks and Changes in the Endowments

We have assumed a fixed fundamental value of the asset s as well as fixed endowments of cash and assets so far. The payoff of the asset in the future can also fluctuate at $t = 1$. Changes in s can be considered aggregate shocks to the economy because they change the return (productivity) of the entire economy.

Proposition 5. *(Aggregate Shock and Vulnerability) The three collateral thresholds $\underline{c}(s,n)$, $\bar{c}(s,n)$, and $c^*(s,D)$ are decreasing in s. Similarly, $\underline{c}(s,n)$ and $\bar{c}(s,n)$ are decreasing in n.*

The result implies that a decrease in the aggregate productivity decreases the safe regions as depicted in Figure 5. For the same collateral ratio, a network such as the complete network might be fully insulated, but after the aggregate shock, the network might become the least stable and resilient network. Thus, an aggregate shock on s can entail different systemic risk levels for the same collateral amount and the same network structure.

Further, we can interpret part of the supply of one asset as agents having additional other assets, which are correlated with the original asset, on their balance sheets. The effect of having these additional assets is analogous to the given results. Declines of the other assets' price p or payoff s would put further downward pressure on the collateral price, and more collateral is needed to mitigate contagion. Thus, this result is in connection to the models with financial constraints in the macro-finance literature.

Similarly, comparative statics with respect to endowments imply the following.

Corollary 1. *The threshold collateral ratios $\bar{c}(s,n)$ and $\underline{c}(s,n)$ are decreasing in e_0 and increasing in h_0. The optimal collateral ratio $c^*(s,D)$ for a given network is decreasing in e_0 and h_0.*

This corollary implies that the collateral ratios required to limit contagion are lower under a greater amount of cash, holding all else equal. The increased amount of cash would provide a buffer to counterparty loss and also reduce the downward pressure on prices from fire sales.

In contrast, $\underline{c}(s,n)$ and $\bar{c}(s,n)$ increases when the supply of assets increases. Although higher h_0 implies more endowments, agents under shocks must sell their assets to pay for the liquidity shock. Since the total amount of cash remains the same, more assets in fire sales would depress the price of assets and the value of collateral. Hence, more collateral is needed to completely prevent contagion.

However, the optimal collateral ratio $c^*(s,D)$ is decreasing in h_0. This is because $c^*(s,D)$ has to be just as large to prevent any further liquidations. For example, $\bar{c}(s,n)$ is set to guarantee $cd\hat{s} \geq d$, so any contract is fulfilled even with zero endowments. The optimal collateral ratio allows for the cash and asset endowments to absorb counterparty losses while making sure that the losses are not enough to trigger additional liquidations. Thus, having more assets would simply mean more indirect buffer to mitigate counterparty exposures by selling the assets to other agents in the network—more indirect links with other agents' net wealth to diversify. This stark difference between the two thresholds shows the importance of tightness of the collateral bounds for the given network structure. This result also highlights a new insight our model can provide to macro-finance models by incorporating interconnectedness.

5.2. Non-trivial Liquidation of Long-term Projects

Almost all the results in this paper can be extended to the case with non-trivial liquidation—that is, $\zeta > 0$. The main difference is that the amount of cash obtained from liquidation of projects now provides a buffer to cash outflows. Hence, there is an intermediate liquidity shock region in which the complete network is still better than the ring network but worse than the δ-connected network. Moreover, there can be price-induced fire sales, as agents may simply liquidate their projects if the asset is cheap enough. The following proposition summarizes this extension.

Proposition 6. *Suppose that $\zeta > 0$, and the collateral ratio is c.*

1. *If $c < \underline{c}(s,n)$ and $\epsilon_* < \epsilon < \epsilon^*$, then the complete network is more stable and resilient than the ring network. Furthermore, if $\epsilon > \epsilon_* + \zeta\xi$, there exists a δ-connected network that is more stable and resilient than the complete network.*

2. *If $c < \underline{c}(s,n)$ and $\epsilon > \epsilon^*$, then there exists d^* such that for any $d > d^*$, the complete and ring networks are the least stable and resilient networks. Furthermore, for a small enough δ, a δ-connected network is more resilient and stable than the complete network.*

3. *If either $\epsilon < \epsilon_*$ or $c \geq \underline{c}(s,n)$ holds, then $p = \hat{s}$, the complete network is the most stable and resilient network, and the ring network is the least stable and resilient network. The amount of liquidation in the ring network decreases as c increases.*

5.3. Other Generalizations

There are other ways to generalize the results. As in Acemoglu et al. (2015), we can consider a model with no collateralizable assets. All the main results of the baseline model hold, as we show in Section D in the appendix. Similarly, allowing for multiple liquidity shocks is relatively straightforward, as in Acemoglu et al. (2015), and, again, most results hold in this extension as we show in Section C.2 in the appendix.

Allowing for heterogeneous collateral ratios is challenging because the difference in payments depends on each individual collateral ratio for different price levels. Thus, there would be many different price regions of contagion depending on the shock size and network structure. Similarly, analytical results with a more general shock distribution would be quite complex because of the exceedingly large number of dimensions to consider. Fortunately, both types of extensions can be solved numerically, as the maximum equilibrium is generically unique.

6. Numerical Analysis

Using numerical simulations, we examine the patterns of contagion of δ-connected networks more closely. These networks exhibit greater social surplus compared to complete and ring networks when facing significant liquidity shocks by containing contagion within one component, when δ values are small. We go beyond this theoretical insight and analyze δ-connected networks further by changing the degree of intercomponent exposures and heterogeneous collateral ratios. Given potential policy concerns regarding the increasing interconnectedness across financial systems, understanding the contagion patterns and social welfare implications of δ-connected networks could be imperative.

In the first set of exercises, we assess the social surplus of different types of δ-connected networks, while varying the interconnection between two components of complete networks as well as the collateral ratio. Specifically, we consider scenarios where the two components are of equal size and where one is smaller than the other. In each scenario, we maintain uniform collateral ratios across all loans.

Let δ be the portion of each agent's total debt obligation to each agent in the alternative component as represented in Figure 6a, where the network is segmented into two even components S and S^c. Each agent owes a large liability of $(d - 10\delta d)/9$ to every agent within the same component and a small liability of δd to every agent in the opposing component. In the heterogeneous case, one component has 5 agents and the other component has 15, as in Figure 6b. Only one agent is shocked at a time, with a large shock $\epsilon > \epsilon^*$, so we consider both cases in which the shocked agent is in the smaller component and in the larger component, respectively.

We examine each network's social surplus and the collateral price, while varying δ, intercomponent exposure, and c, the collateral ratio. Define δ^* as the minimum δ value such that the δ-connected network has the lowest social surplus (i.e. full contagion) when c is just below $\underline{c}$. δ^* indicates up to what degree two financial networks can be interlinked without contagion completely spreading from one to the other.

Under the homogeneous components case, $\delta^* = 0.48\%$ and $\underline{c} = 0.55$, as shown in Figures 7a and 7b. We find that δ^* is relatively small, meaning that it requires little exposure for contagion to spread from one component to the other. Furthermore, contagion is minimized, contained to half the agents, when $\delta < 0.48\%$ but is maximized when $\delta \geq 0.48\%$, exhibiting a quick phase transition. When $c \geq 0.55$, the network becomes safe and fully insulated, with the maximum price.

Under the first heterogeneous case, where the shocked agent is in the larger 15-agent component, $\delta^* = 0.34\%$ and $\underline{c} = 0.55$. As shown in Figures 7c and 7d, the pattern is similar to the homogeneous components case. The 15 agents in the shocked component all liquidate, so the remaining cash is quite low, resulting in low price up to δ^*. Notice that this δ^*, 0.34%, is smaller than the one in the previous case, 0.48%, as it would be easier to "contaminate" 5 agents as opposed to 10.

The second heterogeneous case, where the shocked agent is in the 5-agent component, does not follow the same pattern. As shown in Figure 7e, from $0 \leq \delta < 2\%$, c^* diminishes from 0.69 to $\underline{c}$. This implies that when the collateral ratio is high, above $\underline{c}$, social welfare improves with greater interconnection to a larger component, as greater interconnection would raise payments made to the 5 agents that help fulfill their debt obligations, utilizing the large amount of cash buffer from the 15-agent component. When c increases from 0.42 to $\underline{c}$, δ^* increases until $\delta = 2\%$, meaning that greater interconnection can be sustained without contagion spilling over to the larger component.

In the second set of exercises, we use the δ-connected network with uneven components to explore variations in three representative collateral ratios: c_1, which remains consistent for debt between agents from different components (intercomponent collateral ratio); c_2, which remains consistent for debt within the component containing the shocked agent (within collateral ratio of the shocked); and c_3, which remains consistent for debt between agents in the non-shocked component (within collateral ratio of the non-shocked). Our baseline is the homogeneous collateral ratios case where one agent receives a large liquidity shock of $\epsilon > \epsilon^*$ and δ is δ^*. We analyze how variations in the values of c_1, c_2, and c_3 affect social surplus and collateral asset price, a modest step towards characterizing contagion that involves heterogeneous collateral ratios.

Our results as shown in Figure 8 provide two interesting findings. First, we find that c_3 does not affect social surplus for any given c_1 and c_2. This is because even if agents in the non-shocked component can guarantee their payments to each other (c_3 is very high), they must also fulfill their payments to the agents in the shocked component when c_1 is not high enough. If agents in the shocked component have to sell their assets perhaps as a result of a low c_2, then asset prices must lower, increasing the liquidity needs for the agents in the non-shocked component. Agents in the non-shocked component also have to fire sell, amplifying the liquidity needs further and resulting in full contagion. Hence, the non-shocked component can remain intact only if either intercomponent exposures are limited (c_1 is high enough) or the shocked component remains intact (c_2 is high enough).[15] For this reason, we do not show the dimension of c_3 in the figures, as all the figures remain the same regardless of the values of c_3.

Second, we find interactions between c_1 and c_2 can change the degree of contagion. The most effective collateral ratio in enhancing social surplus is c_2, where complete insulation, even for the shocked component, is achieved when c_2 surpasses certain thresholds. Even when c_2 is below the threshold, an interaction between c_2 and c_1 exists: the lower c_2 is, the higher c_1 needs to be to prevent contagion from spreading to the non-shocked component, as depicted in the lower regions of Figure 8 in a step-like fashion. Interestingly, in the case of the smaller component being shocked, raising the intercomponent collateral ratio can reduce the threshold for the within collateral ratio required for full insulation of the entire network, as illustrated in Figure 8a. This is because the smaller component has fewer agents to diversify

[15]The opposite is also true for low c_3. Even if none of the contracts between the agents in non-shocked component is collateralized ($c_3 = 0$), within-component-payments will be fulfilled as they will net out to each other, as long as outflows of cash to the shocked component is small enough.

the liquidity shock away with cash buffers, thus necessitating a higher c_2 level to maximize social surplus when c_1 is low.

7. Conclusion

This paper constructed a model with both debt and collateral market contagion including endogenous fire-sale prices. Our results highlight the importance of the interaction between the level of collateral and the network structure. Collateral can mitigate social inefficiency from liquidations by guaranteeing the payment in case of borrower default. However, collateral may fail to do so, depending on the network structure and the collateral amount, as the collateral price may quickly fall once contagion accelerates fire sales, and vice versa. The stark difference in collateral prices depending on the collateral ratio and network structure shows the fragility of collateral as a buffer for counterparty exposures. Since the payoff of collateral assets is public and fixed, our results also highlight the importance of liquidity flows in bilateral lending relationships during market stress and fire sales.

Our results highlight both the importance of explicit collateral and the fragility of it. Contrary to the models using the total amount of capital or going-concern value as collateral, contracts in our model specify explicitly designated collateral, which is more in line with how collateralized debt markets operate in the real world. In our model, even if the total amounts of assets or individual asset holdings are the same, changes in collateral ratios can drastically change the contagion pattern. Also, this contagion pattern depends on the interconnectedness of agents.

The model also provides insights on policy for financial stability. We derive the threshold collateral levels to prevent any contagion or contagion through the collateral price channel. In addition, we derive the optimal collateral ratio, the minimum collateral required to prevent any further losses of social surplus by preventing further contagion, for a given network structure. Further, we show how these thresholds change as the aggregate economy changes.

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ϵ	x_{21}	x_{32}	x_{43}	x_{54}	x_{15}	Total Wealth	Fire Sales	Asset Price	Liquidations
2	8	8	8	8	8	8	0	1	1
3	8	8	8	8	8	8	1	1	1
10	1	4	7	8	8	3	3	1	3
15	0	2	4	6	8	0	5	0	5

Table 1: Example with 5 agents under 4 shock regimes

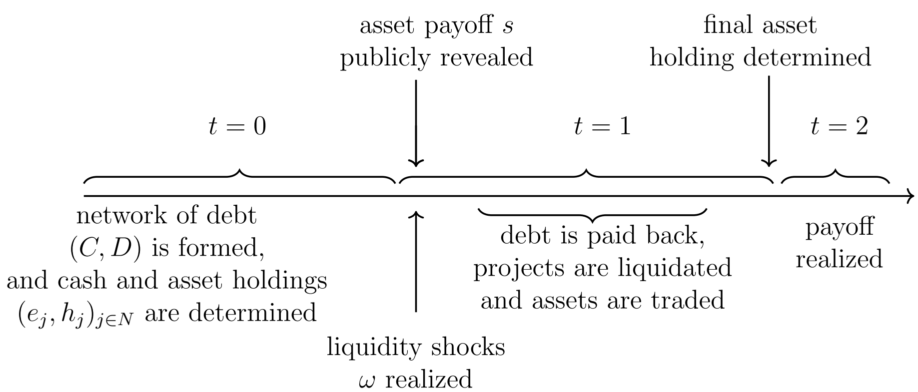


Figure 1: Timeline of the model

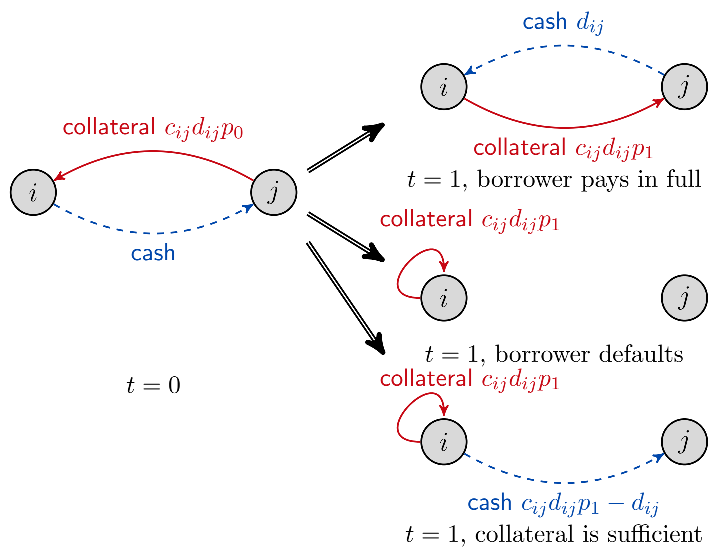


Figure 2: Flows of cash and collateral for three cases

Note: The two nodes, i and j, represent the lender and borrower of a contract, respectively. The blue dashed arrows represent flows of cash, and the red arrows represent flows of collateral. The left figure shows the flows in $t = 0$. The top-right figure shows the flows in the case in which the borrower pays in full in $t = 1$, the middle-right figure shows the flows in the case with borrower default in $t = 1$, and the bottom-right figure shows the flows in the case in which the collateral value exceeds the payment in $t = 1$.

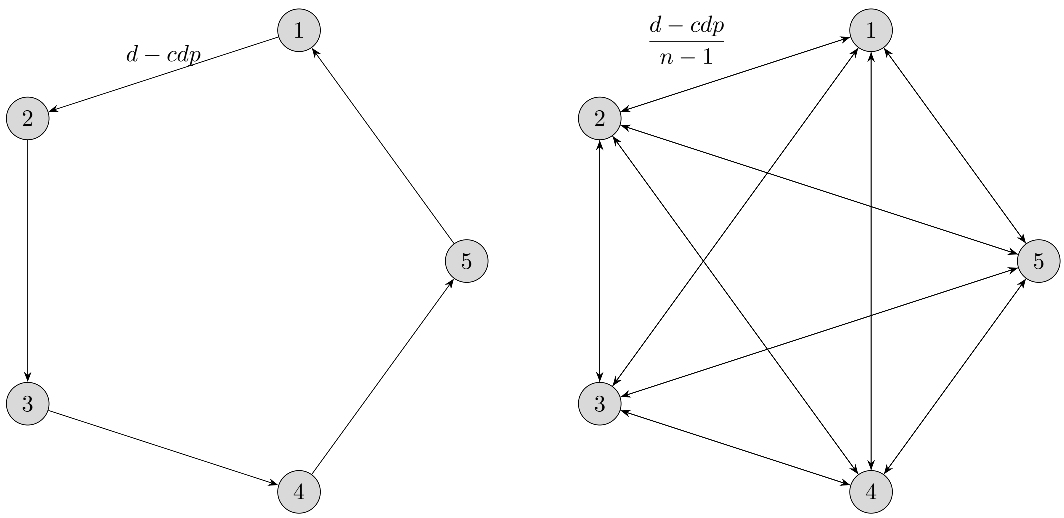


Figure 3: The ring network and the complete network

Note: The total payment obligation after netting the collateral posted is $d - cdp$.

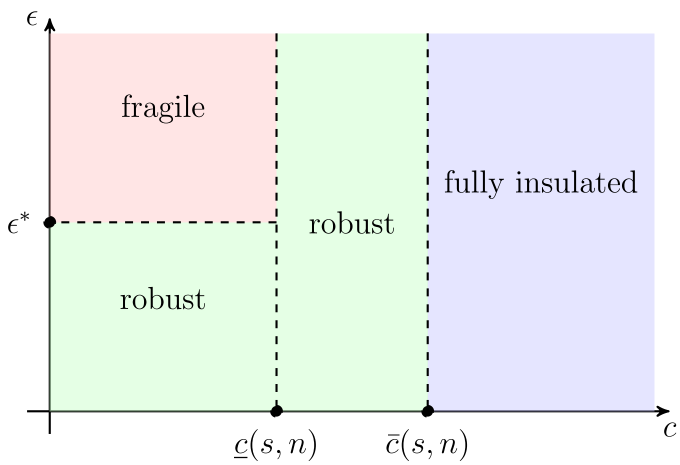


Figure 4: Different regimes depending on the collateral ratio and liquidity shock

Note: The horizontal axis represents the collateral ratio, c, and the vertical axis represents the size of the liquidity shock, ϵ. If c is above the threshold $\bar{c}(s,n)$, then the equilibrium is under the fully insulated regime, which is shaded in blue. If c is above the threshold $\underline{c}(s,n)$ but below $\bar{c}(s,n)$, then the equilibrium is under the robust regime, which is shaded in green, and there is limited collateral contagion. Under the robust regime, having more links would make the network more stable and resilient. If c is below $\underline{c}(s,n)$, then the regimes depend on the size of the liquidity shock. If $\epsilon < \epsilon^*$, then the equilibrium is under the robust regime. However, if $\epsilon > \epsilon^*$, the equilibrium is under the fragile regime, which is shaded in red, and there is active collateral contagion. Under the fragile regime, having more links can make the network less stable and resilient.

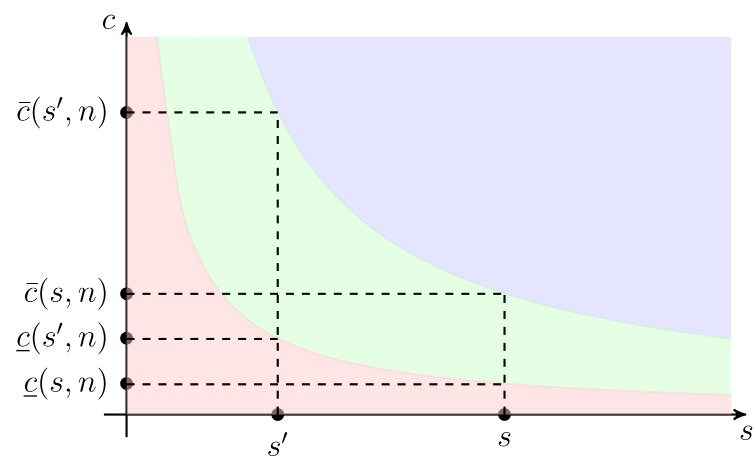
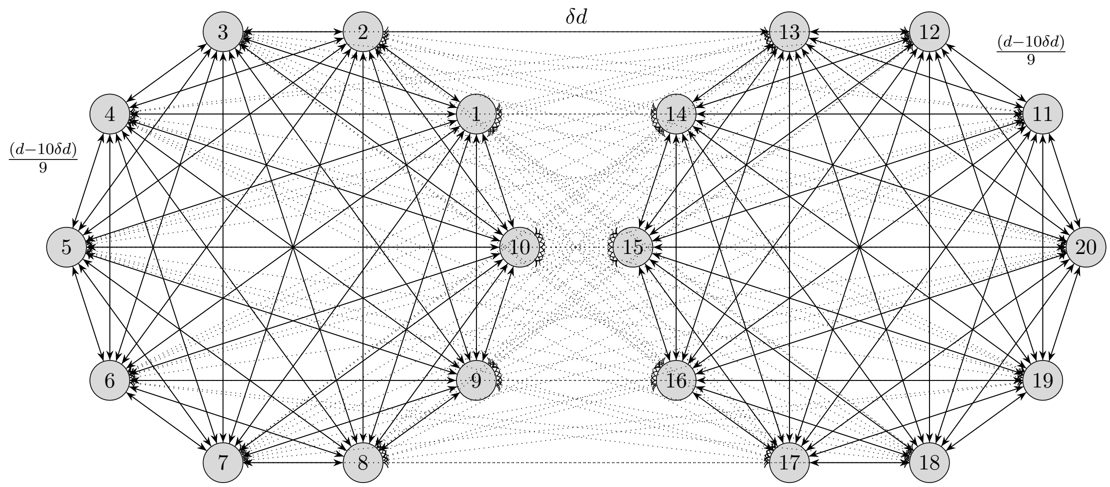
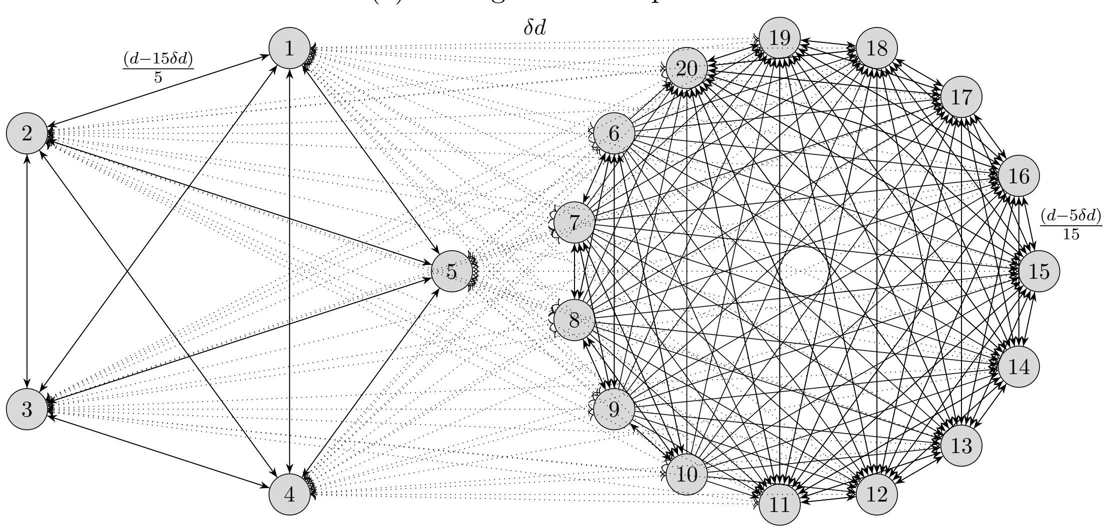


Figure 5: Aggregate shock and vulnerability

Note: The x-axis represents the fair value of the asset, and the y-axis represents the collateral ratio. The red area represents the region with contagion in both debt and collateral, the green area represents the region without collateral contagion, and the blue area represents the region with full insulation by collateral. The figure illustrates the three different regions under the baseline parameters (s, n) and parameters reflecting a negative aggregate shock (s', n) such that $s' < s$. The figure shows that the required collateral ratios to attain the desired level of stability (no collateral contagion or full insulation) increase when there is a negative aggregate shock.

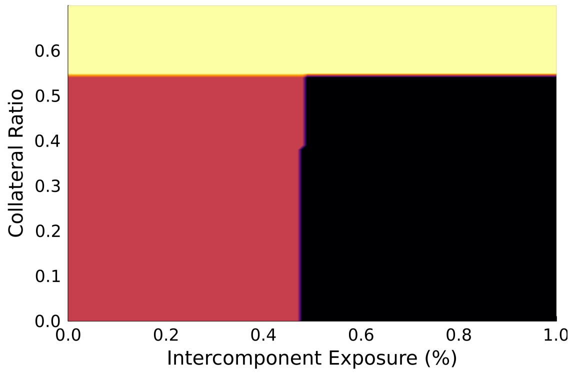


(a) Homogeneous components

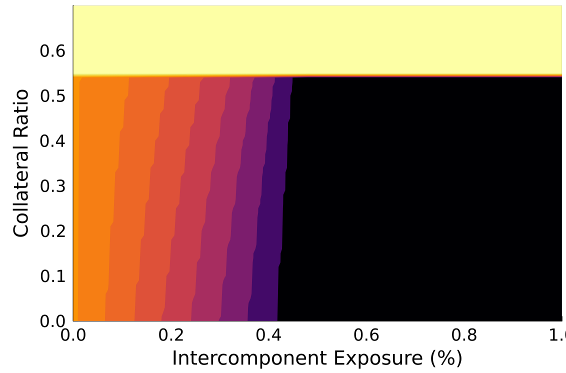


(b) Heterogeneous components

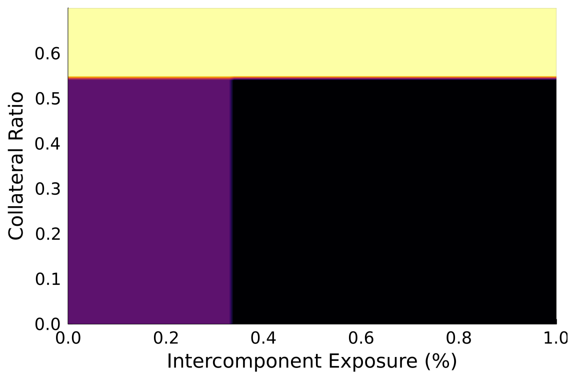
Figure 6: The δ-connected networks for simulations



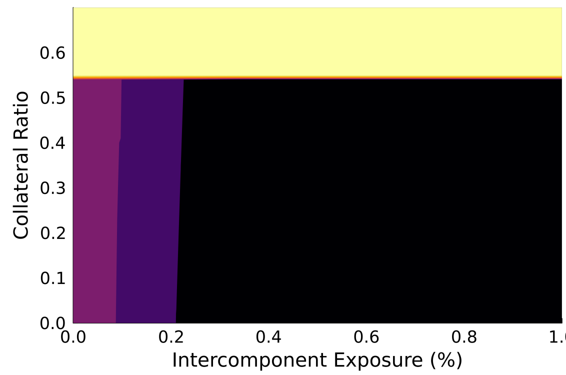
(a) Homogeneous: social surplus



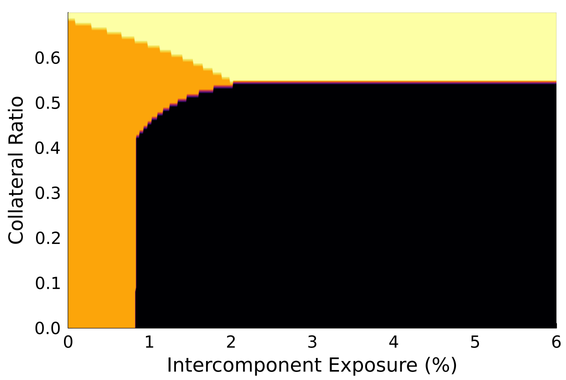
(b) Homogeneous: price



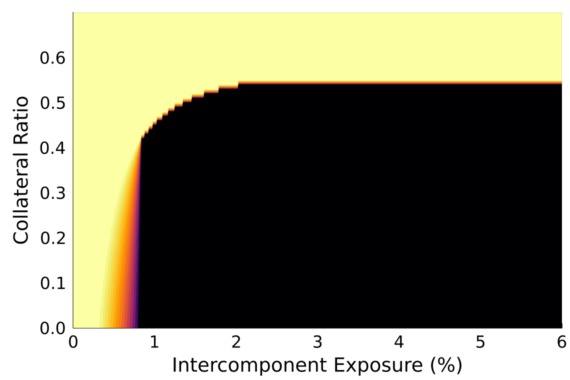
(c) Shocked 15-agent component: social surplus



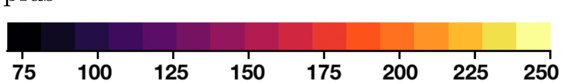
(d) Shocked 15-agent component: price



(e) Shocked 5-agent component: social surplus



(f) Shocked 5-agent component: price



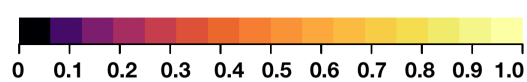
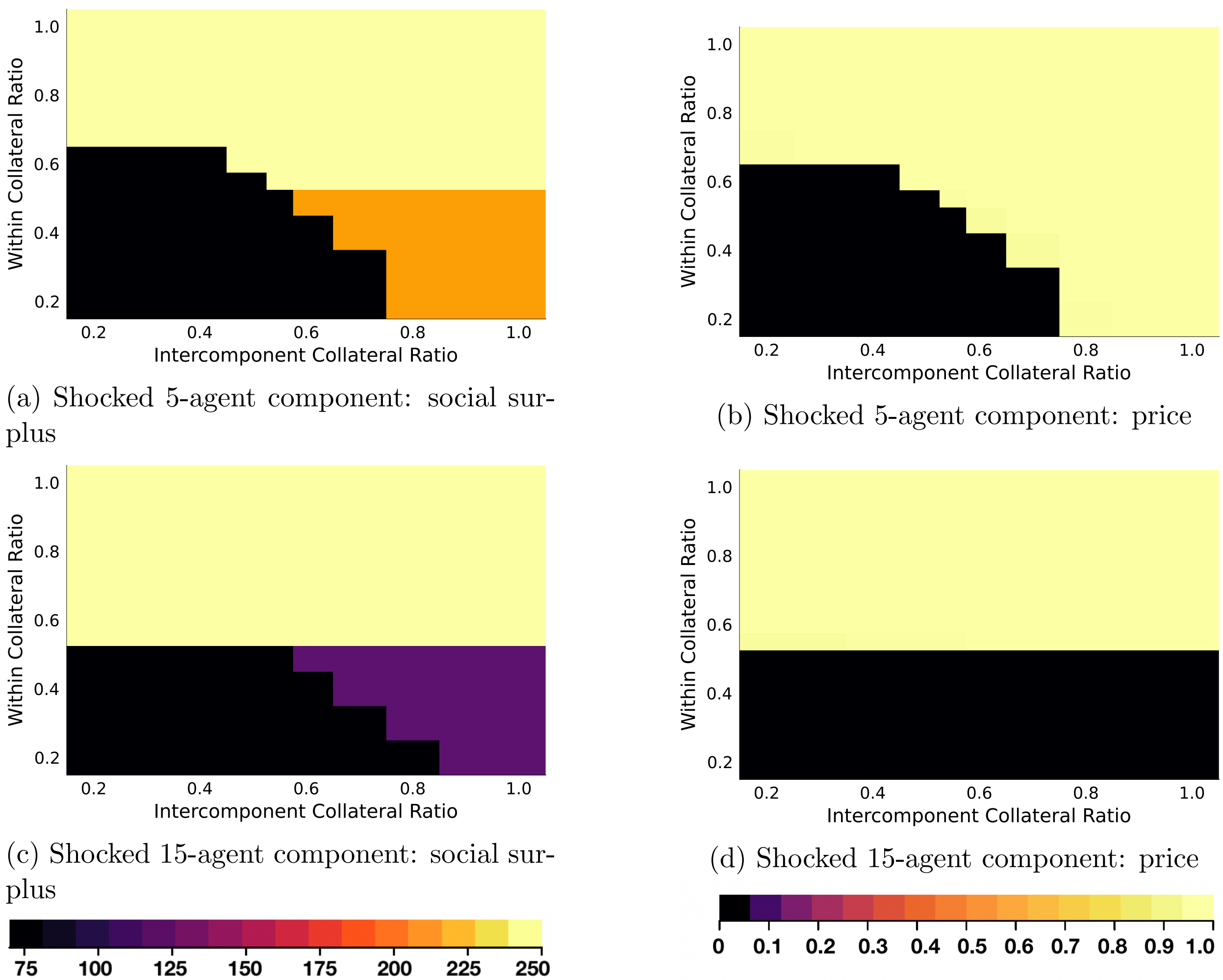


Figure 7: Experiments with δ-connected networks

Note: We evaluate 20 agents, each holding 1 cash endowment (e_0), 2 assets (h_0), and an investment project (ξ) valued at 10. Each agent owes and lends a total debt obligation (d) of $2d^* = 38$. Only 1 agent is under liquidity shock, and the liquidation efficiency (ζ) goes to 0. 1 agent receives a large liquidity shock of $\epsilon > \epsilon^* = 20$.



(a) Shocked 5-agent component: social surplus

(b) Shocked 5-agent component: price

(c) Shocked 15-agent component: social surplus

(d) Shocked 15-agent component: price

Figure 8: Heterogenous collateral ratios in the δ-connected network

Note: We evaluate 20 agents, each holding 1 cash endowment (e_0), 2 assets (h_0), and an investment project (ξ) valued at 10. Each agent owes and lends a total debt obligation (d) of $2d^* = 38$. Only 1 agent is under liquidity shock, and the liquidation efficiency (ζ) goes to 0. 1 agent receives a large liquidity shock of $\epsilon > \epsilon^* = 20$. The x-axis varies the collateral ratio for loans made between two agents in opposing components. The y-axis varies the collateral ratio for loans within the component containing the shocked agent. The figures would remain the same when we change the collateral ratio of the non-shocked component.

Appendix (for online publication only)

A. Preliminaries

A.1. Collateral Netting

We define a useful way of considering network contagion for a given asset price p, which is *collateral netting*. The role of collateral netting is to pre-calculate any overcollateralized payments, as they are guaranteed to be paid in full by the collateral posted. In addition, collateral netting also lumps the collateral of undercollateralized payments into the lender's asset holdings, as the collateral guarantees the market value of collateral even if the borrower pays nothing. Therefore, collateral netting will simplify the derivations while keeping the equilibrium payments the same.

For a given network and environment $(N, C, D, e, h, s, \omega)$ and asset price p, define debt obligations, cash holdings, and asset holdings after collateral netting as the following for any $i, j \in N$:

$$\hat{d}_{ij}(p) = [d_{ij} - c_{ij}d_{ij}p]^+, \tag{13}$$

$$\hat{e}_j(p) = e_j + \sum_{\substack{k \in N \\ c_{jk}p > 1}} d_{jk} - \sum_{\substack{i \in N \\ c_{ij}p > 1}} d_{ij}, \tag{14}$$

$$\hat{h}_j(p) = h_j + \sum_{\substack{k \in N \\ c_{jk}p \leq 1}} c_{jk}d_{jk} - \sum_{\substack{i \in N \\ c_{ij}p \leq 1}} c_{ij}d_{ij}. \tag{15}$$

This collateral netting derives an interim network after netting out the collateral and payments across agents for a given price p. For example, if a contract d_{ij} is overcollateralized, $c_{ij}p > 1$, then $\hat{d}_{ij}(p)$ is 0, because full payment is guaranteed by the collateral posted. Collateral netting calculates the transfer of full payment d_{ij} to lender i, which is included in i's cash holdings $\hat{e}_i(p)$ and $\hat{e}_j(p)$, and the transfer of collateral $c_{ij}d_{ij}$ to borrower j, which is included in $\hat{h}_j(p)$. If a contract d_{jk} is undercollateralized, $c_{jk}p \leq 1$, then k owes j an additional amount of $\hat{d}_{jk}(p) > 0$ on top of the collateral value. Because the market value of collateral does not depend on who owns the collateral, assume that the collateral is kept by lenders for undercollateralized debts, without loss of generality. Then, the collateral will be in j's balance sheet in the asset holdings $\hat{h}_j(p)$. Hence, collateral netting simplifies the cross-agent debt payments by taking care of payments related to collateral and the ownership of collateral, which do not depend on whether a borrower defaults or not.

A.2. Payment Equilibrium under Collateral Netting

In line with the literature, we define the interim equilibrium of our full equilibrium for a given price p (not necessarily an equilibrium price) without fire sales and market clearing conditions—that is, the payment and liquidity decisions, $\{x_{ij}(p)\}$ and $\{l_j(p)\}$, which satisfy payment and liquidation rules, for a given asset price p—as the payment equilibrium of $(N, C, D, e, h, s, \omega)$ and p.

We introduce a matrix notation for the payment equilibrium of a collateral-netting network that corresponds to the payment equilibrium of the original network. Let $Q(p) \in \mathbb{R}^{n\times n}$ be the matrix with its (i, j) element as q_{ij} defined in equation (7). Let $\mathbf{d} = (d_1, d_2, \ldots, d_n)'$ be the vector of agents' total inter-agent liabilities and $\mathbf{l} = (l_1, l_2, \ldots, l_n)'$ be the vector of agents' liquidation decisions. Define

$$\hat{z}_j(p) \equiv \hat{e}_j + \hat{h}_j p - \omega_j \epsilon,$$

and $\hat{z} = (\hat{z}_1, \ldots, \hat{z}_n)'$.

Equations (5) and (6) for every agent can be simplified in matrix notation as the system of equations below:

$$\hat{\mathbf{x}} = \left[\min\left\{Q\hat{\mathbf{x}} + \hat{z} + \zeta\hat{\mathbf{l}}, \hat{\mathbf{d}}\right\}\right]^{+}, \tag{16}$$

$$\hat{\mathbf{l}}(p) = \begin{cases} \left[\min\left\{\frac{1}{\zeta}\left(\hat{\mathbf{d}} - Q\hat{\mathbf{x}} - \hat{z}\right), \xi\mathbb{1}\right\}\right]^{+} & \text{if } p \geq s\zeta \\ \xi\mathbb{1} & \text{if } p < s\zeta, \end{cases} \tag{17}$$

where $\hat{x}_j = \sum \hat{x}_{ij}$, $\hat{\mathbf{x}}$ is the vector of $\hat{x}_j$'s, and $\mathbb{1}$ is a vector of ones for the appropriate dimension. Note that if Q is not defined, then the payments are trivially determined, as agents can pay their debt in full using collateral. The function entry p is omitted here and will be omitted from now on unless necessary for exposition. We define the payment equilibrium of a collateral-netting network, a debt network with no collateral after performing collateral netting on the original network, with these modified payment and liquidation rules.

Definition 6. *For a fixed price p and a collateral-netting network $(N, \hat{D}, \hat{e}, \hat{h}, s, \omega)$, $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{l}})$ is a payment equilibrium if it satisfies* (16) *and* (17).

We now show that the payment equilibrium under the *collateral-netting network* is equivalent to the payment equilibrium under the original network. In other words, if $(\hat{\mathbf{x}}, \hat{\mathbf{l}})$ satisfies the payment rule and the liquidation rule as above, then the corresponding $(\mathbf{x}, \mathbf{l})$ satisfies the payment and liquidation rules for the original network for the given price.

Lemma 2. *The net wealth and payments of agents in the payment equilibrium of the collateral-netting network $(N, \hat{D}, \hat{e}, \hat{h}, s, \omega)$ for a given full equilibrium price p is the same as the net wealth and payments of agents in the payment equilibrium of the original network $(N, C, D, e, h, s, \omega)$ for a given full equilibrium price p.*

Proof. The payment under the collateral-netting network is simplified as below for any

$i, j \in N$:

$$\hat{x}_{ij}(p) = \min\left\{\hat{d}_{ij}, q_{ij}(p)\left[\hat{e}_j + \hat{h}_j p + \zeta l_j(p) - \omega_j \epsilon + \sum_{k \in N} \hat{x}_{jk}\right]^+\right\} \tag{18}$$

$$= \left[\min\left\{\hat{d}_{ij}, q_{ij}\left(\hat{e}_j + \hat{h}_j p + \zeta l_j(p) - \omega_j \epsilon + \sum_{k \in N} \hat{x}_{jk}\right)\right\}\right]^+,$$

where the second equality holds because $\hat{d}_{ij} \geq 0$ for any $i, j \in N$ by (13). If $c_{ij}p \leq 1$ and the net wealth for the second case is the same with $m_j(p)$, then the payments are the same. If $c_{ij}p > 1$, then $\hat{d}_{ij}(p) = 0$, but from $\hat{e}_j(p)$ and $\hat{h}_j(p)$, the payment $d_{ij} - c_{ij}d_{ij}p$ will be subtracted from j's net wealth. Therefore, the payments are equivalent to x_{ij} for any $i, j \in N$ as long as the net wealth is equivalent. The corresponding net wealth is

$$\begin{aligned}
\hat{m}_j(p) \equiv& \hat{e}_j + \hat{h}_j(p)p - \omega_j \epsilon + \zeta l_j(p) + \sum_{k \in N} \hat{x}_{jk} - \sum_{i \in N} \hat{d}_{ij} \\
=& e_j + \sum_{\substack{k \in N \\ c_{jk}p > 1}} d_{jk} - \sum_{\substack{i \in N \\ c_{ij}p > 1}} d_{ij} + h_j p + \sum_{\substack{k \in N \\ c_{jk}p \leq 1}} c_{jk}d_{jk}p - \sum_{\substack{i \in N \\ c_{ij}p \leq 1}} c_{ij}d_{ij}p \\
&+ \sum_{\substack{i \in N \\ c_{ij}p \leq 1}} x_{jk} - \sum_{\substack{i \in N \\ c_{ij}p \leq 1}} (d_{ij} - c_{ij}d_{ij}p) - \omega_j \epsilon + \zeta l_j(p) \\
=& e_j + h_j p - \omega_j \epsilon + \zeta l_j(p) + \sum_{k \in N} c_{jk}d_{jk}p + \sum_{k \in N} x_{jk} - \sum_{i \in N} d_{ij} \\
=& m_j(p),
\end{aligned}$$

which implies the net wealth remains the same as in the original network. ■

Note that collateral netting is trivial when the asset price or collateral ratio is high. Therefore, the payment amount and the market value of the asset holding under the collateral-netting network are increasing in p.

The only thing left to check is the market clearing condition, equation (10), and if the market clearing condition holds, the given payment equilibrium values constitute a full equilibrium. The following lemma, which is a direct application of Lemma B2 in Acemoglu et al.

(2015), further simplifies the computation of the payment equilibrium and full equilibrium.

Lemma 3. *Suppose that p is a price from a full equilibrium. Suppose that $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{l}})$ is from a collateral-netting network of a full equilibrium $(\boldsymbol{x}, \boldsymbol{l}, \boldsymbol{m}, p)$. Then, $\hat{\boldsymbol{x}}$ satisfies*

$$\hat{\boldsymbol{x}} = \left[\min\left\{Q\hat{\boldsymbol{x}} + \hat{z} + \zeta\xi\mathbb{1}, \hat{\boldsymbol{d}}\right\}\right]^{+}. \tag{19}$$

Conversely, if $\hat{\boldsymbol{x}} \in \mathbb{R}^n$ satisfies (19), then there exists $\hat{\boldsymbol{l}} \in [0, \xi]^n$ such that $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{l}})$ is a payment equilibrium and the corresponding $(\boldsymbol{x}, \boldsymbol{l}, \boldsymbol{m}, p)$ is a full equilibrium.

Proof of Lemma 3. Suppose that $(\hat{\mathbf{x}}, \hat{\mathbf{l}})$ is from a full equilibrium and forms a payment equilibrium for the equilibrium price p. First, suppose that $p < s\zeta$. Then, every agent will liquidate its assets regardless of the payments so $\hat{\mathbf{l}} = \xi\mathbb{1}$ and the payment rule satisfies (19). Now suppose that $p \geq s\zeta$. By liquidation rule (17), $\zeta\hat{\mathbf{l}} = \left[\min\left\{\left(\hat{\mathbf{d}} - Q\hat{\mathbf{x}} - \hat{z}\right), \zeta\xi\mathbb{1}\right\}\right]^{+}$, which yields

$$\begin{aligned} Q\hat{\mathbf{x}} + \hat{z} + \zeta\hat{\mathbf{l}} &= \max\left\{Q\hat{\mathbf{x}} + \hat{z}, \min\left\{\hat{\mathbf{d}}, Q\hat{\mathbf{x}} + \hat{z} + \zeta\xi\mathbb{1}\right\}\right\} \\ \Rightarrow \min\left\{\hat{\mathbf{d}}, Q\hat{\mathbf{x}} + \hat{z} + \zeta\hat{\mathbf{l}}\right\} &= \min\left\{\hat{\mathbf{d}}, \max\left\{Q\hat{\mathbf{x}} + \hat{z}, \min\left\{\hat{\mathbf{d}}, Q\hat{\mathbf{x}} + \hat{z} + \zeta\xi\mathbb{1}\right\}\right\}\right\} \\ &= \min\left\{\hat{\mathbf{d}}, Q\hat{\mathbf{x}} + \hat{z} + \zeta\xi\mathbb{1}\right\}. \end{aligned}$$

Thus, $\hat{\mathbf{x}} = [\min\{\hat{\mathbf{d}}, Q\hat{\mathbf{x}} + \hat{z} + \zeta\hat{\mathbf{l}}\}]^{+} = [\min\left\{\hat{\mathbf{d}}, Q\hat{\mathbf{x}} + \hat{z} + \zeta\xi\mathbb{1}\right\}]^{+}$.

Now we consider the other direction. Again, if $p < s\zeta$, then agents will liquidate all of their projects. Therefore, if p is an equilibrium price, then there exists an equilibrium with $(\hat{\mathbf{x}}, \hat{\mathbf{l}})$. Finally, suppose that $p \geq s\zeta$. Then, from (17) and (19), $\hat{\mathbf{l}}(p) = \left[\min\left\{1/\zeta\left(\hat{\mathbf{d}} - Q\hat{\mathbf{x}} - \hat{z}\right), \xi\mathbb{1}\right\}\right]^{+}$ is satisfied. Plugging this expression into the notation of $\mathbf{X}$, we get

$$
\begin{aligned}
Q\hat{\mathbf{x}} + \hat{z} + \zeta\hat{\mathbf{l}} &= \max\left\{Q\hat{\mathbf{x}} + \hat{z}, \min\left\{\hat{\mathbf{d}}, Q\hat{\mathbf{x}} + \hat{z} + \zeta\xi\mathbb{1}\right\}\right\} \\
\Rightarrow \left[\min\left\{\hat{\mathbf{d}}, Q\hat{\mathbf{x}} + \hat{z} + \zeta\hat{\mathbf{l}}\right\}\right]^+ &= \left[\min\left\{\hat{\mathbf{d}}, \max\left\{Q\hat{\mathbf{x}} + \hat{z}, \min\left\{\hat{\mathbf{d}}, Q\hat{\mathbf{x}} + \hat{z} + \zeta\xi\mathbb{1}\right\}\right\}\right\}\right]^+ \\
&= \left[\min\left\{\hat{\mathbf{d}}, Q\hat{\mathbf{x}} + \hat{z} + \zeta\xi\mathbb{1}\right\}\right]^+ = \hat{\mathbf{x}},
\end{aligned}
$$

as in the other direction. Therefore, the equilibrium payment rule is also satisfied, and $(\hat{\mathbf{x}}, \hat{\mathbf{l}})$ is a payment equilibrium for price p. Because p is a full equilibrium price, the corresponding $(\mathbf{x}, \mathbf{l}, \mathbf{m}, p)$ is a full equilibrium. ■

A.3. Useful Lemmas

For the following results and proofs of the results in the main text, we assume that liquidity shocks are randomly received by κ number of agents, so $\omega_j \in \{0, 1\}$ for all $j \in N$, and $\kappa \equiv \sum_{j\in N} \omega_j$. Thus, we generalize the baseline model which assumes $\kappa = 1$.

The following lemma is useful in showing the existence of equilibrium as well as other properties of the equilibrium. The main intuition of the proof is that either assets should be owned by the non-defaulting agents as confiscated collateral or the increase in p increases the cash inflow of defaulting agents, resulting in more payments to the non-defaulting agents.

Lemma 4. *The aggregate positive net wealth* $\sum\limits_{j\in N} [m_j(p)]^+$ *is increasing in the asset price* p. *Moreover, if* $\sum\limits_{j\in N} [m_j(p)]^+ > 0$, *then it is strictly increasing in the asset price* p.

Proof of Lemma 4. First, for the given asset price p, denote the set of agents defaulting as $\mathcal{D}(p)$ and the complement set as $\mathcal{S}(p) \equiv N \backslash \mathcal{D}(p)$. If $\sum\limits_{j\in N} [m_j(p)]^+ = 0$, then it cannot decrease further and it is trivially increasing in p. Thus, suppose that $\sum\limits_{j\in N} [m_j(p)]^+ > 0$.

Recall that

$$\begin{aligned} m_j(p) =& e_j + h_j p + \sum_{k\in N} c_{jk} d_{jk} p + \zeta l_j(p) - \omega_j \epsilon - \sum_{i\in N} d_{ij} + \sum_{k\in N} x_{jk}(p) \\ =& e_j + h_j p + \sum_{k\in \mathcal{D}(p)} c_{jk} d_{jk} p + \zeta l_j(p) - \omega_j \epsilon - \sum_{i\in N} d_{ij} + \sum_{k\in \mathcal{S}(p)} d_{jk} + \sum_{k\in \mathcal{D}(p)} x_{jk}(p). \end{aligned}$$

Summing up the net wealth of non-defaulting agents implies

$$\sum_{j\in\mathcal{S}(p)} m_j(p) = \sum_{j\in\mathcal{S}(p)} \left(e_j + h_j p + \zeta l_j(p) - \omega_j \epsilon - \sum_{i\in\mathcal{D}(p)} d_{ij} \right) + \sum_{j\in\mathcal{S}(p)} \sum_{k\in\mathcal{D}(p)} \left(c_{jk} d_{jk} p + x_{jk}(p) \right).$$

Note that the coefficients of p are positive. Also note that changes in the liquidation amount $l_j(p)$ do not decrease net wealth. If $p < s\zeta$, then $l_j(p) = \xi$, which is non-decreasing in p. If $p \geq s\zeta$, $l_j(p)$ does not decrease net wealth when p increases, because the liquidation amount should cover the discrepancy $b_j(p) - a_j(p)$, if there is any, and the liquidation amount should be just enough to maintain $m_j(p) = 0$. Hence, an increase in price does not decrease the net wealth of non-defaulting agents through the changes in $l_j(p)$. Therefore, $\sum_{j\in\mathcal{S}} m_j(p)$ is strictly increasing in p if

$$\sum_{j\in\mathcal{S}} \sum_{k\in\mathcal{D}} \left(x_{jk}(p) + c_{jk} d_{jk} p \right)$$

is strictly increasing in p. Recall that

$$\begin{aligned} x_{jk}(p) = \min \Bigg\{ & d_{jk} - c_{jk} d_{jk} p, q_{jk}(p) \Bigg[e_k + h_k p + \sum_{l\in N} c_{kl} d_{kl} p - \sum_{l\in N} c_{lk} d_{lk} p \\ & + \sum_{l\in N} [c_{lk} d_{lk} p - d_{lk}]^+ + \sum_{l\in N} x_{kl}(p) + \zeta\xi - \omega_k \epsilon \Bigg]^+ \Bigg\}, \end{aligned}$$

for any $k \in \mathcal{D}(p)$. We will focus on the case in which $\sum_{l\in N} [c_{lk} d_{lk} p - d_{lk}]^+ = 0$, which is for undercollateralized contracts. If $c_{lk} d_{lk} p - d_{lk} > 0$ for $l \neq j$, then $x_{jk}(p)$ will increase even further with the increase in p because an increase in $c_{lk}d_{lk}p - d_{lk}$ increases k's net wealth and payments to others, and the argument below still holds. Also, if $c_{jk}d_{jk}p - d_{jk} > 0$, then $x_{jk}(p) + c_{jk}d_{jk}p = d_{jk}$ will be non-decreasing in p. Therefore, it is enough to show that the statement is true in the case where there is no $l \in N$ such that $c_{lk}d_{lk}p - d_{lk} > 0$ for any $k \in \mathcal{D}(p)$.

Given that, we can simplify the expression as

$$\begin{aligned}
&\sum_{j\in\mathcal{S}}\sum_{k\in\mathcal{D}}(x_{jk}(p) + c_{jk}d_{jk}p) \\
&\quad= \sum_{j\in\mathcal{S}}\sum_{k\in\mathcal{D}}\left(q_{jk}(p)\left[e_k + h_k p + \sum_{l\in N} c_{kl}d_{kl}p - \sum_{l\in N} c_{lk}d_{lk}p + \sum_{l\in N} x_{kl}(p) + \zeta\xi - \omega_k\epsilon\right]^+ + c_{jk}d_{jk}p\right) \\
&\quad= \sum_{j\in\mathcal{S}}\sum_{k\in\mathcal{D}}\left(q_{jk}(p)\left[F_k(p) + \sum_{l\in\mathcal{D}(p)} c_{kl}d_{kl}p + \sum_{l\in\mathcal{D}(p)} x_{kl}(p) - \sum_{l\in N} c_{lk}d_{lk}p\right]^+ + c_{jk}d_{jk}p\right),
\end{aligned} \tag{20}$$

where $F_k(p) = e_k + h_k p + \sum_{l\in\mathcal{S}(p)} d_{kl} + \zeta\xi - \omega_k\epsilon$ is strictly increasing in p.

Case 1. Consider the case in which

$$F_k(p) + \sum_{l\in\mathcal{D}(p)} c_{kl}d_{kl}p + \sum_{l\in\mathcal{D}(p)} x_{kl}(p) - \sum_{l\in N} c_{lk}d_{lk}p > 0, \qquad \forall k \in \mathcal{D}(p)$$

so no agents are defaulting on their liquidity shocks (senior debt). The weights should add up to 1, as $\sum_{j\in N} q_{jk} = 1$; therefore, the payments of defaulting agents should satisfy

$$\begin{aligned}
x_k(p) \equiv \sum_{l\in N} x_{lk}(p) &= \sum_{l\in\mathcal{S}(p)} x_{lk}(p) + \sum_{l\in\mathcal{D}(p)} x_{lk}(p) \\
&= F_k(p) + \sum_{l\in\mathcal{D}(p)} c_{kl}d_{kl}p + \sum_{l\in\mathcal{D}(p)} x_{kl}(p) - \sum_{l\in N} c_{lk}d_{lk}p > 0
\end{aligned}$$

for any $k \in \mathcal{D}(p)$. Thus, (20) can be rearranged as

$$\sum_{j\in\mathcal{S}(p)}\sum_{k\in\mathcal{D}(p)}(x_{jk}(p)+c_{jk}d_{jk}p) = \sum_{k\in\mathcal{D}(p)}F_k(p)+\sum_{k\in\mathcal{D}(p)}\sum_{l\in\mathcal{D}(p)}(c_{kl}d_{kl}p-c_{lk}d_{lk}p)+\sum_{k\in\mathcal{D}(p)}\sum_{l\in\mathcal{D}(p)}x_{kl}(p)$$
$$-\sum_{k\in\mathcal{D}(p)}\sum_{l\in\mathcal{S}(p)}c_{lk}d_{lk}p-\sum_{l\in\mathcal{D}(p)}\sum_{k\in\mathcal{D}(p)}x_{lk}(p)+\sum_{j\in\mathcal{S}(p)}\sum_{k\in\mathcal{D}(p)}c_{jk}d_{jk}p$$
$$=\sum_{k\in\mathcal{D}(p)}F_k(p),$$

which is strictly increasing in p.

Case 2. Now suppose that some agents default on their liquidity shocks. Denote the set of such agents as $\mathcal{B}(p)$, which implies $\forall k \in \mathcal{B}(p)$, $x_{jk}(p) = 0$, for any $j \in N$. We will often omit the argument p for the sets $\mathcal{B}(p), \mathcal{D}(p)$, and $\mathcal{S}(p)$ from now on for notational simplicity. Then, rearranging (20) yields

$$\sum_{j\in\mathcal{S}}\sum_{k\in\mathcal{D}}(x_{jk}(p)+c_{jk}d_{jk}p) = \sum_{k\in\mathcal{D}\backslash\mathcal{B}}F_k(p)+\sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{l\in\mathcal{B}}(c_{kl}d_{kl}p-c_{lk}d_{lk}p)$$
$$-\sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{l\in\mathcal{B}}x_{lk}(p)+\sum_{j\in\mathcal{S}}\sum_{l\in\mathcal{B}}c_{jl}d_{jl}p. \tag{21}$$

Case 2.1. If $x_{lk}(p)$ is zero—that is, $q_{lk}(p) = 0$—for all $l \in \mathcal{B}(p)$ and $k \in \mathcal{D}(p)\backslash\mathcal{B}(p)$, then the right-hand side of (21) is trivially increasing in p by applying collateral constraints twice. The actual steps are similar to the steps shown in a more general case, Case 2.2, below.

Case 2.2. Suppose that $q_{lk}(p) > 0$ for some $l \in \mathcal{B}(p)$ and $k \in \mathcal{D}(p)\backslash\mathcal{B}(p)$. Recall that

$$x_k(p) = F_k(p)+\sum_{l\in\mathcal{D}}c_{kl}d_{kl}p-\sum_{l\in N}c_{lk}d_{lk}p+\sum_{l\in\mathcal{D}\backslash\mathcal{B}}x_{kl}(p)$$

for any $k \in \mathcal{D}(p)\backslash\mathcal{B}(p)$. Therefore, the matrix notation of the aggregate payments from $\mathcal{D}(p)\backslash\mathcal{B}(p)$ becomes

$$x_{\mathcal{D}\backslash\mathcal{B}} = G_{\mathcal{D}\backslash\mathcal{B}} + Q_{\mathcal{D}\mathcal{D}}x_{\mathcal{D}\backslash\mathcal{B}},$$

where $x_{\mathcal{D}\backslash\mathcal{B}}$ is a $|\mathcal{D}(p)\backslash\mathcal{B}(p)| \times 1$ vector of $x_k(p)$ for each $k \in \mathcal{D}(p)\backslash\mathcal{B}(p)$, $G_{\mathcal{D}\backslash\mathcal{B}}$ is a $|\mathcal{D}(p)\backslash\mathcal{B}(p)| \times 1$ vector of $F_k(p) + \sum_{l\in\mathcal{D}(p)} c_{kl}d_{kl}p - \sum_{l\in N} c_{lk}d_{lk}p$ for each $k \in \mathcal{D}(p)\backslash\mathcal{B}(p)$, and $Q_{\mathcal{D}\mathcal{D}}$ is a $|\mathcal{D}(p)\backslash\mathcal{B}(p)| \times |\mathcal{D}(p)\backslash\mathcal{B}(p)|$ matrix of weights $q_{ij}(p)$ for $i, j \in \mathcal{D}(p)\backslash\mathcal{B}(p)$. Note that the spectral radius of $Q_{\mathcal{D}\mathcal{D}}$ is less than 1 by assumption and $(I - Q_{\mathcal{D}\mathcal{D}})^{-1}$ exists by the property of the Neumann series. Hence,

$$x_{\mathcal{D}\backslash\mathcal{B}} = (I - Q_{\mathcal{D}\mathcal{D}})^{-1} G_{\mathcal{D}\backslash\mathcal{B}},$$

and the sum of payments to agents defaulting on senior debt is represented as

$$\sum_{k\in\mathcal{D}\backslash\mathcal{B}} \sum_{l\in\mathcal{B}} x_{lk}(p) = \mathbb{1}' Q_{\mathcal{B}\mathcal{D}} x_{\mathcal{D}\backslash\mathcal{B}} = \mathbb{1}' Q_{\mathcal{B}\mathcal{D}} (I - Q_{\mathcal{D}\mathcal{D}})^{-1} G_{\mathcal{D}\backslash\mathcal{B}},$$

where $Q_{\mathcal{B}\mathcal{D}}$ is a $|\mathcal{B}(p)| \times |\mathcal{D}(p)\backslash\mathcal{B}(p)|$ matrix of weights $q_{lk}(p)$ for $l \in \mathcal{B}(p)$ and $k \in \mathcal{D}(p)\backslash\mathcal{B}(p)$. Since all entries of $Q_{\mathcal{B}\mathcal{D}}$ are also less than 1, there exists $\eta < 1$ such that

$$\sum_{k\in\mathcal{D}\backslash\mathcal{B}} \sum_{l\in\mathcal{B}} x_{lk}(p) = \eta \left[\sum_{k\in\mathcal{D}\backslash\mathcal{B}} F_k(p) + \sum_{k\in\mathcal{D}\backslash\mathcal{B}} \sum_{l\in\mathcal{B}} (c_{kl}d_{kl}p - c_{lk}d_{lk}p) - \sum_{k\in\mathcal{D}\backslash\mathcal{B}} \sum_{j\in\mathcal{S}} c_{jk}d_{jk}p \right].$$

Thus, (21) implies

$$\begin{aligned} \sum_{j\in\mathcal{S}} \sum_{k\in\mathcal{D}} (x_{jk}(p) + c_{jk}d_{jk}p) =& (1-\eta) \left[\sum_{k\in\mathcal{D}\backslash\mathcal{B}} F_k(p) + \sum_{k\in\mathcal{D}\backslash\mathcal{B}} \sum_{l\in\mathcal{B}} (c_{kl}d_{kl}p - c_{lk}d_{lk}p) \right] \\ &+ \eta \sum_{k\in\mathcal{D}\backslash\mathcal{B}} \sum_{j\in\mathcal{S}} c_{jk}d_{jk}p + \sum_{j\in\mathcal{S}} \sum_{l\in\mathcal{B}} c_{jl}d_{jl}p. \end{aligned}$$

Adding $\sum_{j\in\mathcal{S}} h_j p$ from $\sum_{j\in\mathcal{S}} F_j p$ to the right-hand side makes the coefficient on p as follows:

$$\sum_{j\in\mathcal{S}} h_j + \sum_{j\in\mathcal{S}} \sum_{l\in\mathcal{B}} c_{jl}d_{jl} + \eta \sum_{k\in\mathcal{D}\backslash\mathcal{B}} \sum_{j\in\mathcal{S}} c_{jk}d_{jk} + (1-\eta) \left[\sum_{k\in\mathcal{D}\backslash\mathcal{B}} h_k + \sum_{k\in\mathcal{D}\backslash\mathcal{B}} \sum_{l\in\mathcal{B}} (c_{kl}d_{kl} - c_{lk}d_{lk}) \right], \tag{22}$$

which is again positive by applying collateral constraints twice, as we show in the following. From the collateral constraints for $k \in \mathcal{D}(p)\backslash\mathcal{B}(p)$, we have

$$\begin{aligned}
&\sum_{k\in\mathcal{D}\backslash\mathcal{B}} h_k + \sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{l\in\mathcal{B}} c_{kl}d_{kl} + \sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{k'\in\mathcal{D}\backslash\mathcal{B}} c_{kk'}d_{kk'} + \sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{j\in\mathcal{S}} c_{kj}d_{kj} \\
&\geq \sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{l\in\mathcal{B}} c_{lk}d_{lk} + \sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{k'\in\mathcal{D}\backslash\mathcal{B}} c_{kk'}d_{kk'} + \sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{j\in\mathcal{S}} c_{jk}d_{jk} \\
&\Rightarrow \sum_{k\in\mathcal{D}\backslash\mathcal{B}} h_k + \sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{l\in\mathcal{B}} c_{kl}d_{kl} - \sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{l\in\mathcal{B}} c_{lk}d_{lk} \geq \sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{j\in\mathcal{S}} c_{jk}d_{jk} - \sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{j\in\mathcal{S}} c_{kj}d_{kj}. \qquad (23)
\end{aligned}$$

Similarly, from the collateral constraints for $j \in \mathcal{S}(p)$, we have

$$\begin{aligned}
&\sum_{j\in\mathcal{S}} h_j + \sum_{j\in\mathcal{S}}\sum_{l\in\mathcal{B}} c_{jl}d_{jl} + \sum_{j\in\mathcal{S}}\sum_{k\in\mathcal{D}\backslash\mathcal{B}} c_{jk}d_{jk} + \sum_{j\in\mathcal{S}}\sum_{j'\in\mathcal{S}} c_{jj'}d_{jj'} \\
&\geq \sum_{j\in\mathcal{S}}\sum_{l\in\mathcal{B}} c_{lj}d_{lj} + \sum_{j\in\mathcal{S}}\sum_{k\in\mathcal{D}\backslash\mathcal{B}} c_{kj}d_{kj} + \sum_{j\in\mathcal{S}}\sum_{j'\in\mathcal{S}} c_{j'j}d_{j'j} \\
&\Rightarrow \sum_{j\in\mathcal{S}} h_j + \sum_{j\in\mathcal{S}}\sum_{l\in\mathcal{B}} c_{jl}d_{jl} \geq \sum_{j\in\mathcal{S}}\sum_{l\in\mathcal{B}} c_{lj}d_{lj} + \sum_{j\in\mathcal{S}}\sum_{k\in\mathcal{D}\backslash\mathcal{B}} c_{kj}d_{kj} - \sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{j\in\mathcal{S}} c_{jk}d_{jk}. \qquad (24)
\end{aligned}$$

Hence, plugging (23) and (24) into (22) implies

$$\begin{aligned}
&\sum_{j\in\mathcal{S}} h_j + \sum_{j\in\mathcal{S}}\sum_{l\in\mathcal{B}} c_{jl}d_{jl} + \eta \sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{j\in\mathcal{S}} c_{jk}d_{jk} + (1-\eta)\left[\sum_{k\in\mathcal{D}\backslash\mathcal{B}} h_k + \sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{l\in\mathcal{B}} (c_{kl}d_{kl} - c_{lk}d_{lk})\right] \\
&\geq \sum_{j\in\mathcal{S}}\sum_{l\in\mathcal{B}} c_{lj}d_{lj} + \sum_{j\in\mathcal{S}}\sum_{k\in\mathcal{D}\backslash\mathcal{B}} c_{kj}d_{kj} - (1-\eta)\sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{j\in\mathcal{S}} c_{jk}d_{jk} \\
&\quad + (1-\eta)\left[\sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{j\in\mathcal{S}} c_{jk}d_{jk} - \sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{j\in\mathcal{S}} c_{kj}d_{kj}\right] \\
&= \sum_{j\in\mathcal{S}}\sum_{l\in\mathcal{B}} c_{lj}d_{lj} + \eta \sum_{j\in\mathcal{S}}\sum_{k\in\mathcal{D}\backslash\mathcal{B}} c_{kj}d_{kj} > 0,
\end{aligned}$$

and thus the coefficient on p is positive, implying that the aggregate positive net wealth is strictly increasing in p.

Finally, we consider the changes in the set of defaulting agents with an increase in p.

Because the aggregate net wealth is increasing in p, as in Case 1, an increase in p will only weakly decrease the number of defaulting agents. Therefore, there are no agents in $j \in \mathcal{S}(p)$ who will default due to an increase in p, and there can be agents $j \in \mathcal{D}(p)$ who will be solvent under higher p and their net wealth would be added to the sum of $\sum_{j \in \mathcal{S}(p)} m_j(p)$, increasing the aggregate positive net wealth further. ■

The following lemma, which is Lemma B1 from Acemoglu et al. (2015), is used for the proof of Proposition 1.

Lemma 5. *Suppose that $\beta > 0$. Then,*

$$\left| [\min\{\alpha, \beta\}]^+ - [\min\{\hat{\alpha}, \beta\}]^+ \right| \leq |\alpha - \hat{\alpha}|. \tag{25}$$

Furthermore, the inequality is tight only if either $\alpha = \hat{\alpha}$ or $\alpha, \hat{\alpha} \in [0, \beta]$.

Proof of Lemma 5. **Case 1.** Suppose that $\beta < \alpha, \hat{\alpha}$. Then, inequality (25) becomes

$$0 \leq |\alpha - \hat{\alpha}|,$$

and the inequality is tight only if $\alpha = \hat{\alpha}$.

Case 2. Suppose that $0 \leq \alpha, \hat{\alpha} \leq \beta$. Then, inequality (25) becomes

$$|\alpha - \hat{\alpha}| = |\alpha - \hat{\alpha}|.$$

Therefore, the inequality is always tight if $\alpha, \hat{\alpha} \in [0, \beta]$.

Case 3. Suppose that either $\alpha < 0 \leq \hat{\alpha}$ or $\hat{\alpha} < 0 \leq \alpha$ holds. Then, the left-hand side of (25) is either $|\hat{\alpha}|$ or $|\alpha|$, which is less than the right-hand side of (25), $|\alpha - \hat{\alpha}|$, and the inequality is tight only if $\alpha = \hat{\alpha} < 0$.

Case 4. Suppose that $\alpha, \hat{\alpha} < 0$. Then, inequality (25) becomes

$$0 \leq |\alpha - \hat{\alpha}|,$$

which is tight only if $\alpha = \hat{\alpha}$.

Case 5. Suppose that either $\hat{\alpha} \leq \beta < \alpha$ or $\alpha \leq \beta < \hat{\alpha}$ holds. Then, the left-hand side of (25) is either $|\beta - \hat{\alpha}|$ or $|\alpha - \beta|$, which is less than the right-hand side of (25), $|\alpha - \hat{\alpha}|$, and the inequality can never be tight as $\alpha \neq \hat{\alpha}$. ■

The following lemma is useful for simplifying the market clearing condition.

Lemma 6. *The market clearing asset price can be represented as*

$$p = \min \left\{ \frac{\sum_{j \in N} [m_j(p)]^+}{\sum_{j \in N} h_j}, s \right\}. \tag{26}$$

Proof of Lemma 6. Suppose that $p < s$ makes the market clear. Rearranging the first line of equation (10) yields

$$\begin{aligned}
\sum_{j \notin \mathcal{D}(p)} [m_j(p) - h_j p]^+ &= \sum_{j \in N} \min \left\{ [h_j p - m_j(p)]^+, h_j p \right\} \\
\sum_{j \notin \mathcal{D}(p)} (m_j(p) - h_j p) &= \sum_{j \in \mathcal{D}(p)} h_j p \\
\sum_{j \in N} [m_j(p)]^+ &= \sum_{j \in N} h_j p \\
p &= \frac{\sum_{j \in N} [m_j(p)]^+}{\sum_{j \in N} h_j},
\end{aligned}$$

which holds because $\phi_i(p) = h_i p$ for any $i \in \mathcal{D}(p)$.

Now suppose that $p = s$ satisfies the second line of equation 10, which implies

$$\begin{aligned}\sum_{j \notin \mathcal{D}(s)} [m_j(s) - h_j s]^+ &\geq \sum_{j \in N} \min \left\{ [h_j s - m_j(s)]^+, h_j s \right\} \\ \sum_{j \notin \mathcal{D}(s)} (m_j(s) - h_s p) &\geq \sum_{j \in \mathcal{D}(s)} h_j s \\ \sum_{j \in N} [m_j(s)]^+ &\geq \sum_{j \in N} h_j s \\ s &\leq \frac{\sum_{j \in N} [m_j(s)]^+}{\sum_{j \in N} h_j}.\end{aligned}$$

Combining the two cases implies that the market price is bounded by s or the ratio between the aggregate positive net wealth and the total supply of assets; therefore, (26). ■

The following lemma describes the role of the size of the liquidity shock in determining the upper and lower bounds of the number of defaults, as well as the role of collateral.

Lemma 7. *For a full equilibrium under the assumptions in Section 4, define* $\epsilon^* \equiv \dfrac{ne_0 + n\zeta\xi}{\kappa}$. *Then, the following statements are true:*

1. *If* $\epsilon < \epsilon^*$, *at least one agent does not default.*

2. *If* $\epsilon > \epsilon^*$ *and* $cp < 1$, *at least one agent defaults and cannot even pay the liquidity shock.*

3. *If* $cp \geq 1$, *no agent defaults on inter-agent debt.*

Proof of Lemma 7. For the first statement, suppose $\epsilon < \epsilon^*$ and use the collateral-netting network for the given equilibrium price. Suppose all agents default. Then, the only possible equilibrium price is $p = 0$ by (10). Because every agent defaults,

$$\hat{z}_j + \zeta\xi + \sum_{k \in N} \hat{x}_{jk} \leq \sum_{i \in N} \hat{x}_{ij}$$

for all $j \in N$. However, summing over all $j \in N$ yields

$$ne_0 + n\zeta\xi - \kappa\epsilon \leq 0,$$

which is a contradiction to $\kappa\epsilon < \kappa\epsilon^* = ne_0 + n\zeta\xi$.

For the second statement, suppose $\epsilon > \epsilon^*$ and $cp < 1$ and no one defaults. Then,

$$\hat{z}_j + \zeta\xi + \sum_{k \in N} \hat{x}_{jk} \geq \sum_{i \in N} \hat{x}_{ij}$$

for all $j \in N$. However, summing over all the equations yields

$$n(e_0 + h_0 p + \zeta\xi) - \kappa\epsilon \geq 0, \tag{27}$$

and the only way to satisfy the inequality is for p to be large enough. However, because $ne_0 + n\zeta\xi < \kappa\epsilon$, there will be no cash in the market to clear the market with $p > 0$, as

$$nh_0 p = n(e_0 + h_0 p + \zeta\xi) - \kappa\epsilon$$
$$\Rightarrow 0 = ne_0 + n\zeta\xi - \kappa\epsilon < 0,$$

where the last inequality comes from $\epsilon > \epsilon^*$, so p becomes zero, and the above inequality (27) becomes

$$ne_0 + n\zeta\xi - \kappa\epsilon \geq 0,$$

which is a contradiction to $\epsilon > \epsilon^*$ again.

For the third statement, recall that the payment under the collateral-netting network is

$$\hat{d}_{ij}(p) = [d_{ij} - c_{ij} d_{ij} p]^+ = 0$$

and any payment is fully covered by collateral. ■

The following lemma shows that the upper bound of the number of defaulting agents is decreasing in the collateral ratio c for a given equilibrium price p.

Lemma 8. *Suppose that κ number of agents are under a liquidity shock. Let $\mathcal{D}$ and p be the set of agents defaulting on their inter-agent debt and the price in full equilibrium, respectively, and $cp < 1$. Then, the number of defaults is bounded above and below as the following:*

$$\kappa \leq |\mathcal{D}| < \frac{\kappa \min\{\epsilon, e_0 + h_0 p + d - cdp + \zeta\xi\}}{e_0 + h_0 p + \zeta\xi}.$$

Therefore, the upper bound of the number of defaults is decreasing in the collateral ratio c and the equilibrium asset price p.

Proof of Lemma 8. The lower bound is trivial because the size of the shock causes the agent under the liquidity shock to default regardless, as collateral does not cover the debt obligations by $cp < 1$. Now, consider the upper bound. For the collateral-netting network, recall $\mathcal{D}(p)$ is the set of agents that default under price p. Then, for each agent $j \in \mathcal{D}(p)$,

$$\sum_{i \in N} \hat{x}_{ij} = \left[e_0 + h_0 p + \sum_{k \in N} \hat{x}_{jk} - \omega_j \epsilon + \zeta\xi \right]^+,$$

where the agent can pay a positive amount only if the agent has enough cash inflows to cover the liquidity shock, and payments are zero otherwise. Note that the maximum payment amount an agent can receive as a lender is $d - cdp$. Therefore,

$$\sum_{i \in N} \hat{x}_{ij} = \left[e_0 + h_0 p + \sum_{k \in N} \hat{x}_{jk} - \omega_j \min\{\epsilon, e_0 + h_0 p + d - cdp + \zeta\xi\} + \zeta\xi \right]^+.$$

Summing over all defaulting agents yields

$$\sum_{j \in \mathcal{D}(p)} \left[(e_0 + h_0 p + \zeta\xi) + \sum_{k \in N} \hat{x}_{jk} - \omega_j \min\{\epsilon, e_0 + h_0 p + d - cdp + \zeta\xi\} \right]^+ = \sum_{j \in \mathcal{D}(p)} \sum_{i \in N} \hat{x}_{ij}.$$

Because agents without liquidity shocks cannot have negative net wealth,

$$\sum_{j\in\mathcal{D}(p)}(e_0+h_0p+\zeta\xi)+\sum_{j\in\mathcal{D}(p)}\sum_{k\in N}\hat{x}_{jk}\leq\sum_{j\in\mathcal{D}(p)}\sum_{i\in N}\hat{x}_{ij}+\kappa\min\left\{\epsilon,e_0+h_0p+d-cdp+\zeta\xi\right\},\tag{28}$$

and by canceling out the payments among defaulting agents, we obtain the bound as

$$\kappa\min\left\{\epsilon,e_0+h_0p+d-cdp+\zeta\xi\right\}-(e_0+h_0p+\zeta\xi)|\mathcal{D}(p)| \geq\sum_{i\notin\mathcal{D}(p)}\sum_{j\in\mathcal{D}(p)}(\hat{d}_{ij}-\hat{x}_{ij})>0,$$

where the last inequality comes from the definition of the defaulting agents. Then, rearranging the inequality results in

$$|\mathcal{D}(p)|<\frac{\kappa\min\left\{\epsilon,e_0+h_0p+d-cdp+\zeta\xi\right\}}{e_0+h_0p+\zeta\xi},\tag{29}$$

which is decreasing in the asset price p and collateral ratio c. ■

The more interesting part of the lemma is the upper bound of defaults. The numerator represents the total liquidity outflow from the system. If the liquidity shock is small, then an agent can pay this liquidity shock with the total inflow of cash for that agent. Then, the total outflow from the system is simply the size of the shock, ϵ. However, if the shock is large, the maximum total cash inflow, $e_0+h_0p+d-cdp+\zeta\xi$, will be drained from the system. The denominator represents the individual endowment of each agent. Therefore, if the total outflow from the system can be covered by individual endowments of defaulting agents, then there will be no further defaults. Hence, we obtain the upper bound of the total number of defaulting agents.

Lemma 8 highlights the relationship between the individual endowments $(e_0+h_0p+\zeta\xi)$, the total debt amount (d), and the value of collateral (cdp). The upper bound is decreasing in the endowments, as having more endowments implies that agents have more cash to pay for the liquidity shortfall. The upper bound is increasing in the total debt amount, as the liquidity shocks can trigger more inter-agent defaults. Finally, the upper bound is decreasing in the total value of the collateral, as the existence of collateral guarantees that at least part of the debt is paid through collateral. Hence, collateral is effectively transferring some amount of liquidity to the lenders in case of default.

B. Agent's Optimization Problem and Its Solution

Agent j would like to maximize long-term profit, π_j at $t = 2$, which is composed of cash holdings, asset holdings multiplied by the asset payoff, and the payoff from the long-term project net of the liquidation amount. The decision variables are how much cash (e) and assets (h) to hold and how much liquidation of the long-term investment project to make (l). All of these decisions are subject to wealth as well as paying off the inter-agent liabilities and liquidity shock, while taking the payments from other agents as given. Hence, agent j solves for the following optimization problem:

$$
\begin{aligned}
& \max_{e,h,l} \pi_j = e + hs + (\xi - l) \\
\text{s.t.} \quad & e + hp = [a_j(p) - b_j(p) + \zeta l]^+ \\
& l \geq \underline{l_j}(p) \equiv \left[\min \left\{ \frac{1}{\zeta} \left(b_j(p) - a_j(p) \right), \xi \right\} \right]^+ \\
& e \geq 0, h \geq 0, l \leq \xi,
\end{aligned} \tag{30}
$$

where the first constraint is the budget constraint, the second is the liquidation constraint to satisfy liabilities, the third and fourth are non-negativity constraints for cash and assets, respectively, and the last is the upper bound of the total liquidation.

There are five possible cases. First, suppose that agent j's available budget is zero even after liquidating the entire long-term project as $a_j(p) - b_j + \zeta\xi \leq 0$. Then, all the constraints are binding, and agent j's portfolio is forced to be $(e, h, l) = (0, 0, \xi)$. Suppose that agent j has some budget available for the rest of the cases. Second, suppose that the asset price is $p = s$. Then, agent j is indifferent between holding more cash and holding more assets. Hence, j will divide the budget into any arbitrary combination of e and h. Third, suppose that the asset price is $s\zeta < p < s$. Then, agent j would prefer to buy more assets than cash because the return of buying an asset is s/p, which is greater than the cash return, 1. However, j does not liquidate any long-term project more than the required amount $\underline{l_j}$, as the long-term project return (when it is not liquidated) is $1/\zeta$, which is greater than the asset return, s/p. Fourth, suppose that the asset price is $p < s\zeta$. Then, the asset return is greater than the long-term project return because $s/p > 1/\zeta$. Hence, agent j will liquidate the entire long-term investment project (more than the necessary amount) to buy as much in assets as possible. Fifth, suppose that $p = s\zeta$. Then, agents are indifferent between purchasing more assets, while liquidating the long-term project, and keeping the long-term project. So, j will choose any arbitrary point between the necessary liquidation amount and the full liquidation amount, ξ.

The first-order conditions (FOCs) of the optimization problem are

$$\begin{aligned} \partial e : &\qquad 1 - \lambda_w + \lambda_e = 0, \\ \partial h : &\qquad s - \lambda_w p + \lambda_h = 0, \\ \partial l : &\qquad -1 + \lambda_w \zeta + \lambda_{\underline{l}} - \lambda_{\bar{l}} = 0, \end{aligned}$$

where $\lambda_w, \lambda_e, \lambda_h, \lambda_{\underline{l}}$, and $\lambda_{\bar{l}}$ are the Lagrangian multipliers for the budget constraint, non-negativity constraint for e, non-negativity constraint for h, liquidation constraint for l, and the upper bound constraint for l, respectively.

Complementary slackness conditions are

$$\begin{aligned} &\lambda_{\underline{l}}(l - \underline{l}) = 0, \\ &\lambda_e e = 0, \\ &\lambda_h h = 0, \\ &\lambda_{\bar{l}}(\xi - l_j) = 0, \end{aligned}$$

where $\underline{l} = \underline{l_j}(p)$.

Case 1. $\underline{l} = \xi$. This implies agent j is obligated to liquidate the long-term investment project in full to pay their payment obligations. Also, j does not have extra cash, as $a_j(p) -$

$b_j(p) \leq \xi$. The budget constraint becomes

$$e + hp = 0,$$

and by $e, h \geq 0$ and $p \geq 0$, $(e, h, l) = (0, 0, \xi)$.

Case 2. $\underline{l} < \xi$. Under this case, only partial or no liquidation is required.

Case 2.1. $l_j = \xi$, $e > 0$, $h > 0$. FOCs for e and l imply

$$\lambda_w = 1,$$
$$\lambda_{\bar{l}} = -1 + \lambda_w \zeta,$$

and combining the two yields $\lambda_{\bar{l}} = -1 + \zeta$, which contradicts $\lambda_{\bar{l}} > 0$ because $\zeta < 1$, so this case does not exist.

Case 2.2. $l_j = \xi$, $e > 0$, $h = 0$. FOCs imply

$$\lambda_w = 1,$$
$$\lambda_h = \lambda_w p - s,$$
$$\lambda_{\bar{l}} = -1 + \lambda_w \zeta,$$

implying $\lambda_{\bar{l}} = -1 + \zeta$, which contradicts $\lambda_{\bar{l}} > 0$ because $\zeta < 1$.

Case 2.3. $l_j = \xi$, $e = 0$, $h > 0$. FOCs imply

$$\lambda_w = \frac{s}{p},$$
$$\lambda_e = \frac{s}{p} - 1,$$
$$\lambda_{\bar{l}} = -1 + \zeta \frac{s}{p},$$

which hold only if $p \leq s\zeta$.

Case 2.4. $\underline{l} < l_j \leq \xi$, $e = 0$, $h = 0$. This case does not exist because it violates the budget constraint.

Case 2.5. $l_j = \underline{l}$, $e > 0$, $h = 0$. FOCs imply

$$\lambda_w = 1,$$
$$\lambda_h = p - s,$$
$$\lambda_{\underline{l}} = 1 - \zeta,$$

which hold only when $p \geq s$ and $\underline{l} = 0$ because otherwise agent j's budget is 0, implying $e = 0$.

Case 2.6. $l_j = \underline{l}$, $e = 0$, $h > 0$. FOCs imply

$$\lambda_w = \frac{s}{p},$$
$$\lambda_e = \frac{s}{p} - 1,$$
$$\lambda_{\underline{l}} = 1 - \zeta\frac{s}{p},$$

which hold only if $p \geq s\zeta$ and $\underline{l} = 0$ because otherwise agent j's budget is 0, implying $h = 0$.

Case 2.7. $l_j = \underline{l}$, $e > 0$, $h > 0$. FOCs imply

$$\lambda_w = 1 = \frac{s}{p},$$
$$\lambda_{\underline{l}} = 1 - \zeta,$$

which hold only if $s = p$ and $\underline{l} = 0$ because otherwise agent j's budget is 0, implying $e = h = 0$.

Case 2.8. $l_j = \underline{l}$, $e = 0$, $h = 0$. FOCs imply

$$\lambda_w = \frac{1}{\zeta}(1 - \lambda_{\underline{l}}) = 1 + \lambda_e = \frac{1}{p}(s + \lambda_h),$$

which happens only if $\frac{1}{\zeta} > \frac{s}{p}$ and $\underline{l} = \frac{b-a}{\zeta} < \xi$.

Case 2.9. $\underline{l} < l_j < \xi$, $e > 0$, $h > 0$. FOCs imply

$$\lambda_w = 1 = \frac{s}{p} = \frac{1}{\zeta},$$

which contradicts $\zeta < 1$, implying such case does not exist.

Case 2.10. $\underline{l} < l_j < \xi$, $e = 0$, $h > 0$. FOCs imply

$$\lambda_w = \lambda_e + 1 = \frac{s}{p} = \frac{1}{\zeta},$$

which hold only if $p = s\zeta$.

Case 2.11. $\underline{l} < l_j < \xi$, $e > 0$, $h = 0$. FOCs imply

$$\lambda_w = 1 = \frac{1}{\zeta},$$
$$\lambda_h = p - s,$$

which contradict $\zeta < 1$, so this case does not exist.

To summarize, agent j will liquidate only up to the required amount of the long-term project, $l = \underline{l}$, if $p > s\zeta$; be indifferent between cash and asset holdings only if $p = s$; buy assets using all the available budget if $p < s$ and $p > s\zeta$; liquidate the long-term project in full to buy more assets if $p < s\zeta$; and liquidate an arbitrary amount between $\underline{l}$ and ξ if $p = s\zeta$.

The solution to the optimization problem pins down the liquidation rule and the demand for assets with the market clearing condition in Section 3.

C. Omitted Proofs

C.1. Full Equilibrium

Proof of Lemma 1. Suppose that (X, l, m, p) is a full equilibrium. Then, any of the cash and assets will generate the given payoff to the whole economy, so $\sum_{i \in N}(e_i + h_i s)$ should be part of the welfare. However, the total long-term projects $n\xi$ may not remain intact, as some or all of the projects can be liquidated by l_j amount for each $j \in N$. The total liquidation amount will be $\sum_{i \in N} l_i$ while the cost of early liquidation is $(1-\zeta)$, as only the ζ proportion will be salvaged. Therefore, the social surplus of the economy is $U = \sum_{i \in N}(e_i + h_i s + \xi) - (1-\zeta)l_i$.
■

Proof of Proposition 1.

The first step, which is based on the proof of Proposition 1 in Acemoglu et al. (2015), is to show that there exists a payment equilibrium that is generically unique for any given p. The second step is to show that there exists an equilibrium price p that satisfies the market clearing condition for the given payment and liquidation vectors of the corresponding payment equilibrium under p.

Existence of the payment equilibrium.

First, fix an asset price p. By Lemma 3, it is sufficient to show that there exists $\mathbf{x}^* \in \mathbb{R}^n_+$ that satisfies $\mathbf{x}^* = \left[\min\left\{Q\mathbf{x}^* + \hat{z} + \zeta\xi\mathbb{1}, \hat{\mathbf{d}}\right\}\right]^+$. Define the mapping $\Phi : \mathcal{X} \to \mathcal{X}$ as

$$\Phi(\mathbf{x}) = \left[\min\left\{Q\mathbf{x} + \hat{z} + \zeta\xi\mathbb{1}, \hat{\mathbf{d}}\right\}\right]^+,$$

where $\mathcal{X} = \Pi_{i=0}^{n}[0, d_i]$. This mapping is continuous, and its domain, which is the same as its range, is a convex and compact subset of the Euclidean space. Thus, there exists $\mathbf{x}^* \in \mathcal{X}$ such that $\Phi(\mathbf{x}^*) = \mathbf{x}^*$ by the Brouwer fixed-point theorem. The corresponding $\mathbf{l}^*$ can be obtained, and the pair $(\mathbf{x}^*, \mathbf{l}^*)$ satisfies the payment and liquidation rules in the original network for any given price p.

Generic uniqueness of the payment equilibrium.

Assume that the financial network is connected without loss of generality, as we can apply the proposition for each component of a network that is not connected. Suppose that for the same equilibrium price p, there exist two distinct payment equilibria (X, l) and $(\tilde{X}, \tilde{l})$ such that $X \neq \tilde{X}$. Then, payments from each equilibrium should satisfy (19). Hence, for each agent j,

$$\begin{aligned} |\hat{x}_j - \hat{\tilde{x}}_j| &= \left| \left[\min \left\{ (Q\hat{\mathbf{x}})_j + \hat{z}_j + \zeta\xi, \hat{d}_j \right\} \right]^+ - \left[\min \left\{ \left(Q\hat{\tilde{\mathbf{x}}} \right)_j + \hat{z}_j + \zeta\xi, \hat{d}_j \right\} \right]^+ \right| \\ &\leq \left| (Q\hat{\mathbf{x}})_j - \left(Q\hat{\tilde{\mathbf{x}}} \right)_j \right|, \end{aligned}$$

where the last inequality is coming from the fact that both terms have the same upper bound and from the triangle inequality. Taking the L^1 norm for the vector representation of both sides of the above inequality becomes

$$\begin{aligned} ||\hat{\mathbf{x}} - \hat{\tilde{\mathbf{x}}}|| &\leq \left| \left| Q \left(\hat{\mathbf{x}} - \hat{\tilde{\mathbf{x}}} \right) \right| \right| \\ &\leq ||Q|| \cdot \left| \left| \left(\hat{\mathbf{x}} - \hat{\tilde{\mathbf{x}}} \right) \right| \right| \\ &= ||\hat{\mathbf{x}} - \hat{\tilde{\mathbf{x}}}||, \end{aligned}$$

because Q is column stochastic from the weighting rule (7). Therefore, all the inequalities are binding and

$$|\hat{x}_j - \hat{\tilde{x}}_j| = \left| (Q\hat{\mathbf{x}})_j - \left(Q\hat{\tilde{\mathbf{x}}} \right)_j \right|$$

holds. Because $\hat{\mathbf{x}} = \left[\min \left\{ \hat{\mathbf{d}}, Q\hat{\mathbf{x}} + \hat{z} + \zeta\xi\mathbb{1} \right\} \right]^+$ and by Lemma 5, either

$$(Q\hat{\mathbf{x}})_j = \left(Q\hat{\tilde{\mathbf{x}}} \right)_j$$

or

$$\begin{aligned} 0 &\leq (Q\hat{\mathbf{x}})_j + \hat{z}_j + \zeta\xi \leq d_j \\ 0 &\leq \left(Q\hat{\hat{\mathbf{x}}}\right)_j + \hat{z}_j + \zeta\xi \leq d_j. \end{aligned} \tag{31}$$

Therefore, the set of defaulting agents, $\mathcal{D}(p)$, is the same for the two different payment equilibria. For any agent satisfying (31)—that is, if $j \in \mathcal{D}(p)$—

$$(Q\hat{\mathbf{x}})_j - \left(Q\hat{\hat{\mathbf{x}}}\right)_j = \hat{x}_j - \hat{\hat{x}}_j.$$

For the other case, for all $j \notin \mathcal{D}(p)$, the other equality, $(Q\hat{\mathbf{x}})_j = \left(Q\hat{\hat{\mathbf{x}}}\right)_j$, should hold. Because the collateral-netting network eliminates any idiosyncratic collateral ratio, the payment and weighting matrices are invariant to any permutation. Denote $\underline{Q}$ and $\underline{\hat{x}}$ as the weighting matrix and payment vector for the collateral-netting matrix after a permutation of the order of agents by having $j \in \mathcal{D}(p)$ first and then $i \notin \mathcal{D}(p)$ later. Therefore,

$$\underline{Q}\left(\underline{\hat{\mathbf{x}}} - \underline{\hat{\hat{\mathbf{x}}}}\right) = \begin{bmatrix} \underline{\hat{\mathbf{x}}}_{\mathcal{D}} - \underline{\hat{\hat{\mathbf{x}}}}_{\mathcal{D}} \\ 0 \end{bmatrix},$$

where $\underline{x}_{\mathcal{D}}$ is the subvector of $\underline{x}$ including only the agents in $\mathcal{D}(p)$ and

$$\left|\left|\underline{Q}\left(\underline{\hat{\mathbf{x}}} - \underline{\hat{\hat{\mathbf{x}}}}\right)\right|\right| = ||\underline{\hat{\mathbf{x}}}_{\mathcal{D}} - \underline{\hat{\hat{\mathbf{x}}}}_{\mathcal{D}}||.$$

Thus, $\hat{x}_j = \hat{\hat{x}}_j$ for any $j \notin \mathcal{D}(p)$ and

$$\underline{Q}_{\mathcal{D}}(\underline{\hat{\mathbf{x}}}_{\mathcal{D}} - \underline{\hat{\hat{\mathbf{x}}}}_{\mathcal{D}}) = \underline{\hat{\mathbf{x}}}_{\mathcal{D}} - \underline{\hat{\hat{\mathbf{x}}}}_{\mathcal{D}}, \tag{32}$$

where $\underline{Q}_{\mathcal{D}}$ is the submatrix of $\underline{Q}$ for the agents in $\mathcal{D}(p)$ and $\underline{\hat{\mathbf{x}}}_{\mathcal{D}}$ is a subvector of $\underline{\hat{\mathbf{x}}}$ for the agents in $\mathcal{D}(p)$. If the debt network is connected, then $\underline{Q}$ and $\underline{Q}_{\mathcal{D}}$ are irreducible nonnegative matrices by construction. Then, by the Perron-Frobenius theorem, there exist a simple eigenvalue and right eigenvector whose components are all positive (Gaubert and Gunawardena, 2004).

If $\mathcal{D}(p)$ is a proper subset of N, then all of the column sums are less than one, and the spectral radii for Q and $\underline{Q}_{\mathcal{D}}$ are less than one. This result is due to $\lim_{k\to\infty} ||Q^k|| = 0$, which implies $0 = \lim_{k\to\infty} Q^k\mathbf{v} = \lim_{k\to\infty} \lambda^k\mathbf{v} = \mathbf{v}\lim_{k\to\infty}\lambda^k$, which leads to $\lim_{k\to\infty}\lambda^k = 0$, where λ and $\mathbf{v}$ are the eigenvalue and eigenvector, respectively. All of the eigenvalues of $\underline{Q}_{\mathcal{D}}$ have an absolute value less than one, and $\underline{Q}_{\mathcal{D}}\mathbf{v} = \mathbf{v}$ does not have a non-trivial solution. Hence, (32) cannot hold unless $\mathcal{D}(p) = N$. Then, $\hat{x}_j = (Q\hat{\mathbf{x}})_j + \hat{z}_j + \zeta\xi$ for all $j \in \mathcal{D}(p) = N$ and

$$\begin{aligned}\sum_{j\in N}\hat{x}_j &= \sum_{j\in N}\sum_{i\in N} q_{ij}\hat{x}_i + \sum_{j\in N}\hat{z}_j + n\zeta\xi \\ &= \sum_{i\in N}\hat{x}_i + \sum_{j\in N}\hat{z}_j + n\zeta\xi.\end{aligned}$$

Furthermore, the only asset price p under $\mathcal{D}(p) = N$ is $p = 0$ from (10). In other words, there cannot be multiple equilibria if the equilibrium price is $p > 0$. Finally, even if all the agents default and $p = 0$, the last equation implies

$$\sum_{j\in N}\hat{z}_j(p) = \sum_{j\in N}(e_j - \omega_j\epsilon) = -n\zeta\xi,$$

which holds only for a non-generic set of parameters, n, e, ω, ζ, ξ, and ϵ, which is a line over a multidimensional Euclidean space. Thus, for the given equilibrium price p, the payment and liquidation pair (X, l) is generically unique.

Existence of the full equilibrium.

Now the only condition left for an equilibrium is the market clearing condition for price p. Suppose that there exists a unique (X, l) pair that satisfies the two equilibrium conditions for any given price $p \in [0, s]$.[16] If the resulting payment equilibrium $(\mathbf{x}^*, \mathbf{l}^*)$ generates m that

[16] This is true for any price $p > 0$, as shown in the proof of the generic uniqueness of the payment

satisfies the second line of equation (10), then $(X^*, \mathbf{l}^*, m^*, \phi^*, s)$ is a full equilibrium.

Now suppose the contrary, and only a price $p < s$ makes the market clear. From Lemma 6, we have

$$p = \frac{\sum_{j \in N} [m_j(p)]^+}{\sum_{j \in N} h_j}, \tag{33}$$

and by Lemma 4, the aggregate positive net wealth is continuously (and strictly) increasing in p (as long as $\sum_{j \in N} [m_j(p)]^+ > 0$). Therefore, the numerator of the right-hand side of (33) is continuously increasing in p. Also, $\sum_{j \in N} m_j(p)$ is increasing in p, as shown in the proof of Lemma 4. Thus, $\mathcal{D}(p) \subset \mathcal{D}(p')$ for any $p > p'$, and an increase in p will increase the price even further by including more agents on the numerator.

Define the mapping $\Psi : [0, s] \to [0, s]$

$$\Psi(p) = \frac{\sum_{j \in N} \left[m_j^*(p)\right]^+}{\sum_{j \in N} h_j},$$

where $m_j^*(p)$ is the corresponding net wealth for agent j under $(X^*, \mathbf{l}^*)$, which are derived from the corresponding payments and liquidation amounts after collateral netting for a given price p. Because $\Psi(p)$ is a continuously (and strictly) increasing function of p from $[0, s]$ to $[0, s]$ (in the region $\mathcal{P}$ such that for any $p \in \mathcal{P}$, $0 < \Psi(p) < s$), there exists a fixed point p^*, which is an equilibrium price. Therefore, a full equilibrium $(X^*, \mathbf{l}^*, m^*, \phi^*, p^*)$ exists, and there exists a maximum price $\bar{p}$, which is a full equilibrium price greater than any other full equilibrium prices. ■

equilibrium.

C.2. Contagion and Systemic Risk

For ease of notation, we will often omit the arguments of the functions $\bar{c}(s,n)$, $\underline{c}(s,n)$, and $c^*(s,D)$. Furthermore, we assume that liquidity shocks are randomly received by κ number of agents, so $\omega_j \in \{0,1\}$ for all $j \in N$, and $\kappa \equiv \sum_{j\in N}\omega_j$. Thus, we generalize the baseline model, which assumes $\kappa = 1$.

Proof of Proposition 2.

1. For the proof of the first statement, we will prove a slightly more general version of the statement, which is the following proposition.

Proposition 7. *(Collateral Insulation)*
Suppose that $\kappa < n$ number of agents are hit with a liquidity shock. If

$$\bar{c}(s,n) \equiv \frac{1}{\hat{s}} < \frac{1}{s\zeta},$$

where $\hat{s} \equiv \min\left\{s, \dfrac{(n-\kappa)e_0}{\kappa h_0}\right\}$, then, for any $c \geq \bar{c}(s,n)$, any network is the most resilient and stable network for any ϵ.

Proof of Proposition 7. From the assumption on the size of ζ that prevents trivial liquidations of long-term projects, we have

$$\max\left\{\frac{s}{s}, \frac{s\kappa h_0}{(n-\kappa)e_0}\right\} = \frac{s}{\min\left\{s, \dfrac{(n-\kappa)e_0}{\kappa h_0}\right\}} < \frac{1}{\zeta};$$

hence, there is no price-induced liquidation of long-term projects if the asset price is $p = \min\left\{s, \dfrac{(n-\kappa)e_0}{\kappa h_0}\right\}$.

All the payments are covered by the collateral if $cp \geq 1$ by Lemma 7 in Appendix A.3.

If all κ number of agents default on their senior debt, collateral covers all the payments, and no non-defaulting agent is liquidating the long-term project, then the total available cash in the economy is $(n-\kappa)e_0$. Also, as only κ number of agents are out of the market, from (26), we have

$$nh_0 p \le \sum_{j \notin \mathcal{D}} m_j(p) \le (n-\kappa)(e_0 + h_0 p)$$
$$\Rightarrow \kappa h_0 p \le (n-\kappa) e_0,$$

which implies that the relevant amount of fire sales is κh_0. Therefore, the asset price is either the fundamental value s or the aggregate liquidity divided by the total amount of fire sales—that is,

$$p = \min\left\{s, \frac{(n-\kappa)e_0}{\kappa h_0}\right\}.$$

If $s < \dfrac{(n-\kappa)e_0}{\kappa h_0}$ and $p = s$, then $c \ge 1/s$ will satisfy $cp \ge 1$.
If $s \ge \dfrac{(n-\kappa)e_0}{\kappa h_0}$, then $p = \dfrac{(n-\kappa)e_0}{\kappa h_0}$ and $cp \ge 1$ holds if $c \ge c^\dagger \equiv \dfrac{\kappa h_0}{(n-\kappa)e_0}$. Finally, the network should satisfy the resource constraints and $c \ge c^\dagger$; thus,

$$nh_0 \ge cd \ge \frac{\kappa h_0}{(n-\kappa)e_0} d,$$

which is possible only if $n(n-\kappa)e_0 \ge \kappa d$. Then, for $\bar{c}(s,n) = \dfrac{1}{\min\left\{s, \dfrac{(n-\kappa)e_0}{\kappa h_0}\right\}}$, any $c \ge \bar{c}$ will satisfy $cp \ge 1$. By the uniqueness of the (maximum) full equilibrium, this equilibrium is the only (maximum) equilibrium. In this equilibrium, all of the payments are made in full, and there will be no additional defaults. Thus, any network structure has the most stable and resilient results, as the collateral fully insulates any propagation. ■

2. For the second statement, we first show that there will be no contagion through the price channel by the following lemma.

Lemma 9. *Suppose that* $\kappa \in \{1, \ldots, n\}$, $\epsilon_* = (ne_0 + \kappa\zeta\xi)/\kappa$, *and* $\underline{c}(s,n) \equiv \dfrac{d - ((n-\kappa)/\kappa)\, e_0 + h_0 \hat{s}}{d\hat{s}}$. *The equilibrium asset price is always* $p = s$ *irrespective of the* *network structure when*

1. $c < \bar{c}(s,n)$ *and* $\epsilon < \epsilon_*$ *or*

2. $c \geq \underline{c}(s,n)$

for any ϵ*. Finally, the threshold collateral ratio preventing the asset price from going below the fundamental value is (weakly) lower than the threshold collateral ratio for full insulation—that is,* $\underline{c}(s,n) \leq \bar{c}(s,n)$*.*

Proof of Lemma 9.

Case 1. $\epsilon < \epsilon_*$**.** The market clearing condition (26) implies that the aggregate net wealth of non-defaulting agents determines the asset price p. From (26) and (28), the aggregate net wealth of non-defaulting agents is

$$\begin{aligned}\sum_{j\notin\mathcal{D}} m_j(p) \geq & \sum_{j\notin\mathcal{D}}(e_0 + h_0 p + \zeta l_j(p)) - \sum_{i\in N}\sum_{j\in\mathcal{D}}\hat{x}_{ij} + \sum_{j\in\mathcal{D}}(e_0 + h_0 p + \zeta\xi) + \sum_{j\in\mathcal{D}}\sum_{k\in N}\hat{x}_{jk} - \kappa\epsilon \\ = & ne_0 + nh_0 p + \sum_{j\notin\mathcal{D}}\zeta l_j(p) + |\mathcal{D}|\zeta\xi - \kappa\epsilon,\end{aligned}$$

and the market clearing condition becomes

$$p \leq \left[\frac{ne_0 + nh_0 p + \sum_{j\notin\mathcal{D}}\zeta l_j(p) + |\mathcal{D}|\zeta\xi - \kappa\epsilon}{nh_0}\right]^+.$$

Since $\epsilon < \epsilon^* \equiv \dfrac{ne_0 + n\zeta\xi}{\kappa}$, there exist agents with positive net wealth by Lemma 7. Then, the market clearing condition implies

$$\begin{aligned} p \leq & \frac{ne_0 + nh_0 p + \sum_{j\notin\mathcal{D}}\zeta l_j(p) + |\mathcal{D}|\zeta\xi - \kappa\epsilon}{nh_0} \\ \Rightarrow p \leq & p + \frac{ne_0 + \sum_{j\notin\mathcal{D}}\zeta l_j(p) + |\mathcal{D}|\zeta\xi - \kappa\epsilon}{nh_0}, \qquad \forall p > 0.\end{aligned}$$

Hence, the maximum equilibrium would satisfy the market clearing condition with $p = \hat{s}$.

Case 2. $c \geq \underline{c}$**.** Suppose $c \geq \underline{c} \equiv \dfrac{d - ((n-\kappa)/\kappa)\, e_0 + h_0 \hat{s}}{d\hat{s}}$. Also, assume $\epsilon > \epsilon_*$ because the result is trivially true by Case 1 otherwise.

Case 2.1. $\epsilon > \epsilon^*$**.** At least one agent under a liquidity shock defaults, even on the liquidity shock (senior debt) by Lemma 7 in Appendix A.3 and $cp < 1$, implying that all other agents would suffer the total default of the shocked agent in the amount of $\kappa(d - cdp)$ or less. Suppose that the total number of defaulting agents is $k = |\mathcal{D}| < n$. Then, even if all κ agents default on their senior debt, the market clearing condition is

$$(n-k)(e_0 + h_0 p) + \sum_{j \notin \mathcal{D}} \zeta l_j(p) + (k - \kappa)(e_0 + h_0 p + \zeta\xi) - \kappa(d - cdp) \geq n h_0 p$$
$$(n-\kappa)e_0 + \sum_{j \notin \mathcal{D}} \zeta l_j(p) + (k-\kappa)\zeta\xi - \kappa(d - cdp) \geq \kappa h_0 p.$$

We make the left-hand side the smallest possible by excluding any cash from the liquidation of projects as

$$(n-k)(e_0 + h_0 p) + (k-\kappa)(e_0 + h_0 p) - \kappa(d - cdp) \geq n h_0 p \tag{34}$$
$$\Rightarrow \kappa(cd - h_0)p \geq \kappa d - (n-\kappa)e_0,$$

where the left-hand side of the final inequality is increasing in p with $cd > h_0$, which holds when $c \geq \underline{c}$. Therefore, if $c \geq \underline{c}$, $p = \hat{s}$ holds for the market clearing condition. Also, this implies the left-hand side of (34) is positive, implying that there are agents with positive net wealth—that is, there are solvent agents who can purchase the assets in the market. Therefore, the asset price is $p = \hat{s}$ in the (maximum) equilibrium regardless of the network structure.

Case 2.2. $\epsilon_* < \epsilon \leq \epsilon^*$**.** Define

$$\hat{\epsilon}(p) \equiv \min\left\{\epsilon, e_0 + h_0 p + \zeta\xi + (d - cdp)\right\},$$

which implies the maximum amount of liquidity outflow from the network due to receiving a liquidity shock. If $\hat{\epsilon}(p) < \epsilon$ for the equilibrium price p, then we can use the same steps in Case 2.1 to show that $p = s$.

Now suppose $\hat{\epsilon}(p) = \epsilon$. The market clearing condition for the lower bound of the sum of effective net wealth is

$$ne_0 + nh_0 p + \kappa\zeta\xi - \kappa\epsilon \geq nh_0 p.$$

Because $\epsilon < \epsilon_*$, the market clearing condition trivially holds for any p. If $\hat{s}$ is less than $\hat{p}$ such that

$$\hat{\epsilon}(\hat{p}) = e_0 + h_0\hat{p} + \zeta\xi + (d - cd\hat{p}) = \epsilon,$$

then p can increase up to $p = \hat{p}$ because $cd > h_0$ and $c > \underline{c}$. Since $\hat{\epsilon}(p) = e_0 + h_0 p + \zeta\xi + (d - cdp)$ for any $p > \hat{p}$, we can use the same steps in Case 2.1 to show that $p = \hat{s}$. Otherwise, the price can trivially increase up to $p = \hat{s}$, which will be the maximum equilibrium price.

Finally, we show that $\underline{c} \leq \bar{c}$. The value of $\bar{c}$ can be either $1/s$ or $\kappa h_0/(n-\kappa)e_0$.

Case 1. Suppose that $\dfrac{(n-\kappa)e_0}{\kappa h_0} \geq s$; therefore, $\bar{c} = 1/s$. Then, $\underline{c} \leq \bar{c}$ holds because

$$\begin{aligned}
\underline{c} \equiv \frac{d - ((n-\kappa)/\kappa)\, e_0 + h_0 s}{ds} &\leq \frac{1}{s} \equiv \bar{c} \\
d - ((n-\kappa)/\kappa)\, e_0 + h_0 s &\leq d \\
h_0 s &\leq ((n-\kappa)/\kappa)\, e_0 \\
s &\leq \frac{(n-\kappa)e_0}{\kappa h_0},
\end{aligned}$$

which holds by the initial assumption.

Case 2. Suppose that $\frac{(n-\kappa)e_0}{\kappa h_0} < s$; therefore, $\bar{c} = \frac{\kappa h_0}{(n-\kappa)e_0}$. Then, $\underline{c} < \bar{c}$ holds because

$$\begin{aligned}
\underline{c} \equiv \frac{d - ((n-\kappa)/\kappa)\, e_0 + h_0 s}{ds} &< \frac{\kappa h_0}{(n-\kappa)e_0} \equiv \bar{c} \\
d - ((n-\kappa)/\kappa)\, e_0 + h_0 s &< \frac{d\kappa h_0 s}{(n-\kappa)e_0} \\
\left(d - \frac{n-\kappa}{\kappa} e_0\right)(n-\kappa)e_0 &< (d - \frac{n-\kappa}{\kappa} e_0)\kappa h_0 s \\
\Rightarrow \frac{(n-\kappa)e_0}{\kappa h_0} &< s,
\end{aligned}$$

which holds by the initial assumption. ■

With the price being fixed, we show the remaining parts of the second statement.

Case 1. Suppose that $\epsilon < \epsilon_*$. By Lemma 9, $p = \hat{s}$. Without loss of generality, suppose agent 1 is hit by the liquidity shock.

Consider the ring network first. From Lemma 7 and $\epsilon < \epsilon^*$, there exists an agent that does not default. Hence, agent n, who is the farthest away from agent 1, can pay its payment in full by $d - cd\hat{s}$. Then,

$$m_1(\hat{s}) = e_0 + h_0\hat{s} + d - cd\hat{s} + \zeta\xi - \hat{\epsilon} - (d - cd\hat{s}) < 0,$$

where $\hat{\epsilon} \equiv \min\{\epsilon, e_0 + h_0\hat{s} + d - cd\hat{s} + \zeta\xi\}$. There exists an agent $k \leq n$ who is not defaulting, while $k-1$ defaults. Hence, the payment from the last defaulting agent $k-1$ to k is

$$x_{k,k-1} = d - cd\hat{s} + (k-1)(e_0 + h_0\hat{s} + \zeta\xi) - \hat{\epsilon}.$$

Since k is not defaulting, $m_k(\hat{s}) \geq 0$ implies

$$\begin{aligned} e_0 + h_0\hat{s} + \zeta\xi + x_{k,k-1} &\geq d - cd\hat{s} \\ e_0 + h_0\hat{s} + \zeta\xi + d - cd\hat{s} + (k-1)(e_0 + h_0\hat{s} + \zeta\xi) - \hat{\epsilon} &\geq d - cd\hat{s} \\ \Rightarrow k(e_0 + h_0\hat{s} + \zeta\xi) &\geq \hat{\epsilon}, \end{aligned}$$

which can be rearranged as

$$k \geq \frac{\min\{\epsilon, e_0 + h_0\hat{s} + d - cd\hat{s} + \zeta\xi\}}{e_0 + h_0\hat{s} + \zeta\xi}.$$

Therefore, by Lemma 8, $k-1$ is the maximum number of defaulting agents, so the ring network is the least resilient and stable network.

Now consider the complete network. By Lemma 7, there exists an agent that does not default on inter-agent debt. By symmetry of the complete network, all $n-1$ agents do not default. Recall that $\epsilon < \epsilon_* \equiv ne_0 + \zeta\xi$. We show that agents other than agent 1, who is under a liquidity shock, can fulfill their debt payments in full even without liquidating any of the long-term projects. First, the net wealth of agent j is

$$\begin{aligned} m_j(\hat{s}) &= e_0 + h_0\hat{s} + \sum_{k \neq 1,j} x_{jk} + x_{j1} - (d - cd\hat{s}) \\ &= e_0 + h_0\hat{s} + (n-2)\frac{d - cd\hat{s}}{n-1} + \frac{e_0 + h_0\hat{s} + d - cd\hat{s} + \zeta\xi - \hat{\epsilon}}{n-1} - (d - cd\hat{s}) \end{aligned}$$

for any $j \neq 1$. Multiplying the last expression by $(n-1)$ implies

$$ne_0 + \zeta\xi + nh_0\hat{s} - \hat{\epsilon} > ne_0 + \zeta\xi - \hat{\epsilon} \geq 0,$$

where the last inequality comes from $\epsilon < \epsilon_*$. Therefore, the complete network is the most resilient and stable network if $\epsilon < \epsilon_*$.

Case 2. Now consider the case with $c > \underline{c}$ and $\epsilon > \epsilon_*$. The equilibrium asset price is $p = \hat{s}$ by Lemma 9. Therefore, each agent's required debt payment is $d - cd\hat{s}$. Because $c > \underline{c}$,

$$d - cd\hat{s} < d - \underline{c}d\hat{s} = (n-1)e_0 - h_0\hat{s} \leq \sum_{j \notin \mathcal{D}} m_j(\hat{s})$$
$$\Rightarrow \hat{s} < \frac{(n-1)e_0 - d + cd\hat{s}}{h_0},$$

which implies the market clearing condition is satisfied at $p = \hat{s}$. Hence, even if the agent under a liquidity shock does not pay anything to the remaining agents, the remaining agents combined have sufficient cash to pay the debt to the agent under a liquidity shock and purchase the assets on fire sale at $p = \hat{s}$. Because all agents are symmetric in the complete network, all remaining $n-1$ agents can pay their debt in full. Hence, the complete network is the most stable and resilient network.

Now consider the ring network. Agent 1's total inter-agent debt payment is

$$x_1 = e_0 + h_0\hat{s} + d - cd\hat{s} + \zeta\xi - \hat{\epsilon}.$$

Then, agent 2 adds only the endowment, $e_0 + h_0\hat{s} + \zeta\xi$, to the payment to agent 3 on top of the payment received from agent 1. Agent 3 will reuse this payment from agent 2 and add agent 3's own endowment, so the total payment from agent 3 to agent 4 is $2(e_0 + h_0\hat{s} + \zeta\xi) + e_0 + h_0\hat{s} + d - cd\hat{s} + \zeta\xi - \hat{\epsilon}$, and so forth. Then, agent $k+1$ would have total available cash of $k(e_0 + h_0\hat{s} + \zeta\xi) + e_0 + h_0\hat{s} + d - cd\hat{s} + \zeta\xi - \hat{\epsilon}$, and agent $k+1$ is solvent only if

$$k(e_0 + h_0\hat{s} + \zeta\xi) + e_0 + h_0\hat{s} + d - cd\hat{s} + \zeta\xi - \hat{\epsilon} > d - cd\hat{s},$$

which implies

$$k + 1 > \frac{\hat{\epsilon}}{e_0 + h_0 \hat{s} + \zeta \xi},$$

which is above the upper bound of the number of defaults in Lemma 8 and (29), implying k reached the upper bound of the number of defaults when $p = \hat{s}$. Also, note that the lowest k satisfying the above inequality is decreasing in c. Thus, the ring network is the least stable and resilient network, and the number of defaulting agents is decreasing in c.

Given the two results of the ring and complete network, we can use the method of collateral netting and apply Proposition 6 of Acemoglu et al. (2015) to obtain the last result.

3. For the third statement, suppose that $\epsilon > \epsilon^*$, $c < \underline{c}$, and $d > d^* = (n-1)e_0$. First, note that $cp < 1$ because $c < \underline{c} \leq \bar{c}$ with $\bar{c} \min\{s, (n-1)e_0/h_0\} = 1$ and $p \leq \min\{s, (n-1)e_0/h_0\}$. Consider the complete network. The agent under a liquidity shock defaults even on the senior debt (liquidity shock) by Lemma 7 and defaults fully on the debt obligation of $(d - cdp)/(n-1)$ to all other agents. Suppose the contrary, that $n-1$ agents do not default. Then, each agent's net wealth equation should satisfy

$$(n-2)\frac{d - cdp}{n-1} + e_0 + h_0 p - (d - cdp) \geq 0. \tag{35}$$

The market clearing condition is

$$(n-1)e_0 - d + cdp \geq h_0 p, \tag{36}$$

where $p > 0$ by the assumption that there are surviving agents. Because $d > (n-1)e_0$, if $cd < h_0$, then there is no price $p > 0$ that can satisfy the market clearing condition, so $p = 0$. Then, (35) implies $(n-1)e_0 \geq d$, which is a contradiction. Now suppose that $cd > h_0$. This inequality implies (36) becomes

$$(cd - h_0)p \geq d - (n-1)e_0, \tag{37}$$

and the left-hand side is maximized when $p = s$, which is the case for the maximum equilibrium we are focusing on. Because $c < \underline{c} \equiv \dfrac{d-(n-1)e_0 + h_0 s}{ds}$, (37) becomes

$$d - (n-1)e_0 \leq (cd - h_0)s < \underline{c}ds - h_0 s = d - (n-1)e_0,$$

which is a contradiction. Therefore, $p = 0$, and, again, (35) implies $(n-1)e_0 \geq d$, which is a contradiction. Therefore, all agents default, and the complete network is the least resilient and least stable network.

Now consider the ring network such that agent 1 borrows from agent 2, who borrows from agent 3, and so on. Without loss of generality, let agent 1 be the agent under a negative liquidity shock. Again, by Lemma 7, agent 1 defaults the full amount, $d - cdp$. Agent 2 will pay agent 3 only with the endowment $e_0 + h_0 p$, and then agent 3 will reuse this with its own endowment to pay agent 4 in the amount of $2(e_0 + h_0 p)$. Agent 4 will pay agent 5 in the amount of $3(e_0 + h_0 p)$, and so on. In order not to cascade after k length from agent 1,

$$k(e_0 + h_0 p) > d - cdp. \tag{38}$$

If agent $k+1$ pays the debt in full, then all the subsequent agents, $k+2, k+3, \ldots, n$, can pay in full without selling their asset holdings. The total market clearing condition is

$$\begin{aligned}(n-k)(e_0 + h_0 p) - d + cdp + (k-1)(e_0 + h_0 p) &\geq nh_0 p \\ (cd - h_0)p &\geq d - (n-1)e_0, \end{aligned} \tag{39}$$

if $p > 0$, and the left-hand side is maximized at $p = s$. However, because $c < \underline{c}$, (39) evaluated even at the highest $p = s$ implies

$$d - (n-1)e_0 \leq (cd - h_0)s < \underline{c}ds - h_0 s = d - (n-1)e_0,$$

which is a contradiction. Therefore, $p = 0$ is the market clearing price. Then, (38) becomes

$$ke_0 > d,$$

which implies $k \geq n$—that is, the first non-defaulting agent should be at the minimum distance of n or higher from agent 1, which exceeds the total number of agents n. Therefore, all agents default, and the ring network is the least resilient and least stable network.

Finally, consider a δ-connected network with $\delta < \dfrac{e_0}{(n-1)d}$ and the partition of agents sets being $(\mathcal{S}, \mathcal{S}^c)$ such that $d_{ij} \leq \delta d$ for any $i \in \mathcal{S}$ and $j \in \mathcal{S}^c$. Thus, $\sum_{j \notin S} d_{ij} \leq \delta d |\mathcal{S}^c|$ for any $i \in \mathcal{S}$. Therefore, for any $i \in \mathcal{S}$,

$$\sum_{j \in S} (d_{ij} - d_{ij}cp) \geq d - cdp - \delta(d - cdp)|\mathcal{S}^c| \geq d - cdp - e_0,$$

which implies

$$e_0 + h_0 p + \sum_{j \in S} (d_{ij} - d_{ij}cp) \geq e_0 + \sum_{j \in S} (d_{ij} - d_{ij}cp) \geq d - cdp,$$

so agents in $\mathcal{S}$ can fulfill their debt even when all agents in $\mathcal{S}^c$ do not pay any amount to agents in $\mathcal{S}$. Note that the last inequality holds for any given collateral price p. In other words, all agents in $\mathcal{S}$ remain solvent when $\omega_j = 1$ for an agent $j \in \mathcal{S}^c$ regardless of the size of the shock. Therefore, the δ-connected network is more stable and resilient than the complete or ring networks. ■

Proof of Proposition 3. We prove the first statement first. Denote the set of defaulting agents other than the agent under the shock, agent j, as $\mathcal{D}$ and the set of non-defaulting agents as $\mathcal{S}$. First, note that the weights following the weighting rule do not change with the price p under the uniform collateral ratio because

$$q_{ij}(p) = \frac{d_{ij} - d_{ij}cp}{\sum_{k\in N} d_{kj} - d_{kj}cp} = \frac{d_{ij}(1-cp)}{\sum_{k\in N} d_{kj}(1-cp)} = \frac{d_{ij}}{d}.$$

Hence, we can represent the harmonic distance in (11) from agent i to agent j as

$$\mu_{ij} = 1 + \sum_{k\neq j} q_{ik}\mu_{kj}. \tag{40}$$

Using expression (40), the vectors of harmonic distances can be represented as

$$\mu_{dj} = \mathbb{1} + Q_{dd}\mu_{dj} + Q_{ds}\mu_{sj}, \tag{41}$$

$$\mu_{sj} = \mathbb{1} + Q_{sd}\mu_{dj} + Q_{ss}\mu_{sj}, \tag{42}$$

where μ_{dj} is the $|\mathcal{D}|\times 1$ vector of harmonic distances from agents in $\mathcal{D}$ to j, μ_{sj} is the $|\mathcal{S}|\times 1$ vector of harmonic distances from agents in $\mathcal{S}$ to j, and $Q_{dd}, Q_{ds}, Q_{sd}, Q_{ss}$ are matrices of weights of liabilities for agents within $\mathcal{D}$, from $\mathcal{S}$ to $\mathcal{D}$, from $\mathcal{D}$ to $\mathcal{S}$, and within $\mathcal{S}$, respectively. Solving (41) for μ_{dj} and plugging it into (42) implies

$$\begin{aligned}
\mu_{sj} &= \mathbb{1} + Q_{sd}\left[(I-Q_{dd})^{-1}\mathbb{1} + (I-Q_{dd})^{-1}Q_{ds}\mu_{sj}\right] + Q_{ss}\mu_{sj},\\
\mu_{sj} &= \left[I + Q_{sd}(I-Q_{dd})^{-1}\right]\mathbb{1} + Q_{sd}(I-Q_{dd})^{-1}Q_{ds}\mu_{sj} + Q_{ss}\mu_{sj},\\
\left[I + Q_{sd}(I-Q_{dd})^{-1}\right]\mathbb{1} &= \left[I - Q_{ss} - Q_{sd}(I-Q_{dd})^{-1}Q_{ds}\right]\mu_{sj}.
\end{aligned} \tag{43}$$

The vector of total payments for defaulting agents $\mathbf{x}_d$ is determined as

$$\mathbf{x}_d = Q_{dd}\mathbf{x}_d + (d - cdp)Q_{ds}\mathbb{1} + (e_0 + h_0 p)\mathbb{1}$$

for a given price p, which can be solved as

$$\mathbf{x}_d = (I - Q_{dd})^{-1} [(d - cdp)Q_{ds}\mathbb{1} + (e_0 + h_0 p)\mathbb{1}] . \tag{44}$$

The vector of net wealth of non-defaulting agents is

$$m_s = Q_{sd}\mathbf{x}_d + Q_{ss}(d - cdp)\mathbb{1} + (e_0 + h_0 p)\mathbb{1} - (d - cdp)\mathbb{1},$$

where we use the fact that non-defaulting agents are paying their debt in full as $d - cdp$. Plugging the payments in (44) into the net wealth vector implies

$$m_s = (e_0 + h_0 p)\left[I + Q_{sd}(I - Q_{dd})^{-1}\right]\mathbb{1} - (d - cdp)\left[I - Q_{ss} - Q_{sd}(I - Q_{dd})^{-1} Q_{ds}\right]\mathbb{1},$$

which can be simplified as

$$m_s = (e_0 + h_0 p)\left[I + Q_{sd}(I - Q_{dd})^{-1}\right]\mathbb{1} - (d - cdp)G\mathbb{1}, \tag{45}$$

where $G = I - Q_{ss} - Q_{sd}(I - Q_{dd})^{-1} Q_{ds}$ is a non-singular M-matrix because it is a Schur complement[17] of $I - Q_{dd}$ in the M-matrix,

$$M \equiv \begin{bmatrix} I - Q_{ss} & -Q_{sd} \\ -Q_{ds} & I - Q_{dd} \end{bmatrix},$$

based on Acemoglu et al. (2015) (Berman and Plemmons, 1979, p. 159). Combining (43)

[17]For a matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

the Schur complement of D in M is $M/D \equiv A - BD^{-1}C$ (Poole and Boullion, 1974).

and (45) implies

$$m_s = (e_0 + h_0 p) G \mu_{sj} - (d - cdp) G \mathbb{1}. \tag{46}$$

If the market price is $p < s$, p should be a liquidity-constrained price. Hence, the market clearing condition implies

$$\mathbb{1}' m_s = n h_0 p \tag{47}$$

from Lemma 6. Plugging (46) into the market clearing condition (47) yields

$$(e_0 + h_0 p) \mathbb{1}' G \mu_{sj} = (d - cdp) \mathbb{1}' G \mathbb{1} + n h_0 p,$$

which can be rearranged as (12).

We define the threshold harmonic distance, which is a function of equilibrium price p, as

$$\mu^*(p) \equiv \frac{d - cdp}{e_0 + h_0 p},$$

which is decreasing in p. Thus, it will be easier for an agent to have safe harmonic distance from the agent under the shock under high p. By definition, non-defaulting agents should have $m_s \geq 0$, implying

$$(e_0 + h_0 p) G \mu_{sj} \geq (d - cdp) G \mathbb{1},$$

which is derived from (46). This implies

$$G \mu_{sj} \geq \mu^*(p) G \mathbb{1}. \tag{48}$$

Because G is a non-singular M-matrix, there exists G^{-1} such that $G^{-1} \geq 0$ (Poole and

Boullion, 1974). Hence, G^{-1} is element-wise non-negative, implying that (48) yields

$$\mu_{sj} \geq \mu^*(p)\mathbb{1},$$

proving the contrapositive of the second part of the first statement. Thus, if $\mu_{ij} < \mu^*(p)$, then agent i defaults.

For the second statement, suppose that all agents default and $p = 0$. Then, for any $i \neq j$,

$$x_i = e_0 + \sum_{k \neq j} q_{ik} x_k < d,$$

which can be rearranged as

$$\frac{x_i}{e_0} = 1 + \sum_{k \neq j} q_{ik} \frac{x_k}{e_0} < \frac{d}{e_0} = \mu^*(0).$$

Thus, the expression coincides with the harmonic distance equation (11), using $\mu_{ij} = \frac{x_i}{e_0}$. Therefore, $\mu_{ij} < \mu^*(0)$ for all agents $i \neq j$.

For the third statement, suppose that p is the equilibrium price and

$$\mu^*(p) = \frac{d - cdp}{e_0 + h_0 p} < 1.$$

Hence, $d - cdp < e_0 + h_0 p$, implying that agents can pay their liabilities out of their endowments, as c is large enough. Therefore, no other agents default on their inter-agent liabilities.

■

Proof of Proposition 4.

We first show that the equilibrium asset price is $\hat{s}$ by the following lemma.

Lemma 10. *Suppose that κ agents are hit with liquidity shocks with $\epsilon > \epsilon^*$, $d > d^* \equiv (n - \kappa)(e_0 + \zeta\xi)$, $h_0 < cd$, and none of the other $n - \kappa$ agents default. Then, the equilibrium* *price is* $p = \min\left\{s, \frac{(n-\kappa)e_0}{\kappa h_0}\right\}$.

Proof of Lemma 10.

Case 1. Suppose $d - cds > 0$.

Because only κ agents default, the sum of net wealth of the remaining agents is

$$\sum_{j\notin\mathcal{D}} m_j(p) \geq (n-\kappa)(e_0 + h_0 p) - \kappa(d - cdp),$$

where strict inequality holds if $l_j > 0$ for some $j \notin \mathcal{D}$ and $d_{ik} > 0$ for some $i, k \in \mathcal{D}$.

Suppose that the equilibrium price is $p < s$. Then, the market clearing condition implies

$$\frac{(n-\kappa)(e_0 + h_0 p) - \kappa(d - cdp)}{nh_0} \leq \frac{\sum_{j\notin\mathcal{D}} m_j(p)}{nh_0} = p,$$

implying

$$\begin{aligned} (n-\kappa)e_0 + (n-\kappa)h_0 p - \kappa d + \kappa cdp &\leq nh_0 p \\ \Rightarrow \frac{(n-\kappa)e_0 - \kappa d}{\kappa(h_0 - cd)} &\leq p < s, \end{aligned}$$

where $h_0 - cd < 0$. From the last inequality,

$$\begin{aligned} \kappa(h_0 s - cds) &< (n-\kappa)e_0 - \kappa d \\ \kappa(h_0 s - cds) + (n-\kappa)h_0 s + \kappa cds &< (n-\kappa)e_0 - \kappa d + (n-\kappa)h_0 s + \kappa cds \\ nh_0 s &< (n-\kappa)(e_0 + h_0 s) - \kappa(d - cds) \\ s &< \frac{(n-\kappa)(e_0 + h_0 s) - \kappa(d - cds)}{nh_0} \leq \frac{\sum_{j\notin\mathcal{D}} m_j(s)}{nh_0}, \end{aligned}$$

and the last inequality implies that the aggregate positive net wealth evaluated at $p = s$ satisfies the market clearing condition for $p = s$. Thus, the maximum equilibrium price has to be $p = s$ instead of $p < s$.

Case 2. Suppose $d - cds \leq 0$.

Then, there exists $\bar{p}$ such that $d - cd\bar{p} = 0$ and $\bar{p} \leq s$. We show that the equilibrium price should be $p \geq \bar{p}$.

Similar to the previous case, suppose the contrary, $p < \bar{p}$. Using the exact same steps of the previous case, while replacing s with $\bar{p}$, we obtain

$$\begin{aligned} &\frac{(n-\kappa)e_0 - \kappa d}{\kappa(h_0 - cd)} < \bar{p} \\ &\qquad \Rightarrow \bar{p} < \frac{(n-\kappa)(e_0 + h_0\bar{p})}{nh_0} \leq \frac{\sum_{j \notin \mathcal{D}} m_j(\bar{p})}{nh_0}, \end{aligned}$$

where we used $d - cd\bar{p} = 0$ for the last inequality. Therefore, the market clearing condition is satisfied, and the equilibrium price should be $p \geq \bar{p}$. Hence, all contracts are fully covered, and the market clearing condition becomes

$$p = \begin{cases} s & \text{if } \dfrac{(n-\kappa)e_0}{\kappa h_0} > s \\ \dfrac{(n-\kappa)e_0}{\kappa h_0} & \text{otherwise} \end{cases}.$$

Combining the two cases, we have the equilibrium price as $p = \min\left\{s, \dfrac{(n-\kappa)e_0}{\kappa h_0}\right\}$. ■

From the proof of the first statement of Proposition 3 with (48), we know that there exists an equilibrium (in fact, the maximum equilibrium) in which any agent j with $\mu_{ij} \geq \mu^*(p)$, where i is the agent under shock, does not default. Thus, all other agents do not liquidate when $\mu_{ij} \geq \mu^*(p)$ for any j for the given equilibrium price p.

Given the equation for $\mu^*(p)$ from Proposition 3 and Lemma 10, we solve for c^* by setting $p = \hat{s}$ and $\mu^*(\hat{s})$ to the minimum harmonic distance μ_{ij}, $\underline{\mu} = \min_{i,j} \mu_{ij}$, for a given network:

$$\begin{aligned} &\mu^*(\hat{s}) = (d - cd\hat{s})/(e_0 + h_0\hat{s}) = \underline{\mu} \\ &\Rightarrow c^* = \frac{d - \underline{\mu}(e_0 + h_0\hat{s})}{d\hat{s}}. \end{aligned}$$

Therefore, c^* is the minimum collateral ratio that prevents any further liquidation from occurring for the given network. ■

C.3. Extensions

Proof of Proposition 5. First, we show that the cutoff collateral ratio for full insulation, $\bar{c}(s,n)$, is decreasing in s and n. Recall that $\bar{c}(s,n) = \max\left\{\frac{1}{s}, \frac{\kappa h_0}{(n-\kappa)e_0}\right\}$. If $\bar{c}(s,n) = 1/s$, then $\bar{c}(s',n) = 1/s'$, so the cutoff for full insulation is higher at s'—that is, $\bar{c}(s,n) < \bar{c}(s',n)$. Under this case, $\bar{c}(s,n) \leq \bar{c}(s,n')$, as $\bar{c}(s,n')$ is either $1/s$ or $\frac{\kappa h_0}{(n'-\kappa)e_0}$, which is greater than $1/s$. Otherwise, $\bar{c}(s,n) = \bar{c}(s',n) = \frac{\kappa h_0}{(n-\kappa)e_0}$ and $\bar{c}(s,n) = \frac{\kappa h_0}{(n-\kappa)e_0} < \bar{c}(s,n') = \frac{\kappa h_0}{(n'-\kappa)e_0}$ for $n > n'$.

Now we show that $\underline{c}(s,n)$ is decreasing in s and n. Recall that

$$\underline{c} \equiv \frac{d - ((n-\kappa)/\kappa)\, e_0 + h_0 s}{ds},$$

which is trivially decreasing in n. The first-order derivative of $\underline{c}$ with respect to s is

$$\begin{aligned}\frac{\partial \underline{c}}{\partial s} &= \frac{h_0 ds - d\left(d - ((n-\kappa)/\kappa)\, e_0 + h_0 s\right)}{(ds)^2} \\ &= \frac{d\left(\frac{n-\kappa}{\kappa} e_0 - d\right)}{(ds)^2} < 0,\end{aligned}$$

where the last inequality comes from $d > (n-1)e_0 > \frac{n-\kappa}{\kappa} e_0$.

Finally, we show that $c^*(s,D)$ is decreasing in s for a given network D. If $\hat{s} = \frac{(n-\kappa)e_0}{\kappa h_0}$, then the statement is trivially true, as c^* is constant across s. Now consider the case in which $\hat{s} = s$. The partial derivative of $c^*(s,D)$ with respect to s implies

$$\frac{\partial c^*(s,D)}{\partial s} = \frac{-\underline{\mu} h_0 ds - d^2 + d\underline{\mu} e_0 + d\underline{\mu} h_0 s}{(ds)^2} = \frac{d\left(\underline{\mu} e_0 - d\right)}{(ds)^2} < 0,$$

where the last inequality holds because $d > (n-1)e_0 \geq \underline{\mu} e_0$. Hence, $c^*(s,D)$ is decreasing in s. ■

Proof of Corollary 1. From the proof of Proposition 2, we have

$$\underline{c} \equiv \frac{d - ((n-\kappa)/\kappa)\, e_0 + h_0 s}{ds}$$
$$\bar{c} \equiv \max\left\{\frac{1}{s}, \frac{\kappa h_0}{(n-\kappa)e_0}\right\},$$

which are (weakly) decreasing in e_0 and increasing in h_0. From the equation of $c^*(s,D)$, the partial derivatives of c^* are

$$\frac{\partial c^*(s,D)}{\partial e_0} = \frac{-\underline{\mu}}{d\hat{s}} < 0,$$
$$\frac{\partial c^*(s,D)}{\partial h_0} = \frac{-\underline{\mu}\hat{s}}{d\hat{s}} < 0,$$

for e_0 and h_0, respectively. Hence, c^* is decreasing in both e_0 and h_0. ■

Proof of Proposition 6.

1. For the first statement, consider the complete network first. By Lemma 7 and symmetry of the complete network, no agents other than the shocked agent, denoted as agent 1 without loss of generality, default under the complete network. The surviving agents may liquidate their long-term projects either because they are short of liquidity following the liquidation rule or because the price of the asset is below or equal to $s\zeta$. Recall that

$$\hat{\epsilon}(p) = \min\{\epsilon, e_0 + h_0 p + \zeta\xi + d - cdp\},$$
$$p^\dagger = \left[\frac{d + e_0 + \zeta\xi - \epsilon}{cd - h_0}\right]^+.$$

The liquidation rule for the complete network when $p > s\zeta$ is

$$l_j = \left[\frac{\frac{d-cdp}{n-1} - e_0 - h_0 p - \frac{e_0 + h_0 p + \zeta\xi + d - cdp - \hat{\epsilon}(p)}{n-1}}{\zeta}\right]^+,$$

for each $j \notin \mathcal{D}$. Hence, the liquidation rule implies

$$\zeta L_c = [\hat{\epsilon}(p) - ne_0 - nh_0 p - \zeta\xi]^+ + \zeta\xi,$$

where L_c is the sum of liquidation amounts of all agents under the complete network for $p > s\zeta$.

The common market clearing condition for both networks is

$$\begin{aligned} ne_0 + nh_0 p + \zeta L - \hat{\epsilon}(p) &\geq nh_0 p \\ \Rightarrow \zeta L &\geq \hat{\epsilon}(p) - ne_0, \end{aligned} \tag{49}$$

where L denotes the sum of liquidation amounts of all agents. Note that (49) holds with equality if $p < s$ and with strict inequality if $p = s$.

The liquidation rule for the ring network when $p > s\zeta$ is

$$\zeta L_r = \left\lfloor \frac{\hat{\epsilon}}{e_0 + h_0 p + \zeta\xi} \right\rfloor \zeta\xi + \left[\hat{\epsilon} - \left\lfloor \frac{\hat{\epsilon}}{e_0 + h_0 p + \zeta\xi} \right\rfloor (e_0 + h_0 p + \zeta\xi) - (e_0 + h_0 p) \right]^+,$$

which implies that the first $k = \left\lfloor \frac{\epsilon}{e_0 + h_0 p + \zeta\xi} \right\rfloor < n$ agents default and fully liquidate their long-term project to pay to their subsequent agents, and the first non-defaulting agent, $k+1$, may liquidate further if there is a shortage of cash even after exhausting its endowment of $e_0 + h_0 p$.

Case 1. Suppose that $cd \leq h_0$. Then, $p^\dagger = 0$ because $d + e_0 + \zeta\xi > n(e_0 + \zeta\xi) > \epsilon$ and $cd - h_0 \leq 0$. Hence, $\hat{\epsilon}(p) = \epsilon$ for any $p \geq 0$, and (49) becomes

$$\zeta L \geq \epsilon - ne_0.$$

Now consider the incentives to liquidate for each agent. First, consider the complete network. Suppose that $p > s\zeta$, so agents will liquidate only up to the necessary amount. The liquidation rule for the complete network under $p > s\zeta$ implies

$$L_c = \frac{\epsilon - ne_0 - nh_0 p}{\zeta};$$

hence, the market clearing condition becomes

$$-nh_0 p \geq 0,$$

which implies $p = 0$. Hence, any $p > s\zeta$ cannot be an equilibrium price, and $p = s\zeta$ at most. Then, agents are indifferent between liquidating and not liquidating the long-term project. The total liquidation amount should be

$$\zeta L_c = \epsilon - ne_0,$$

which satisfies the market clearing condition. Note that this is feasible, as $0 < L < n\xi$ because $ne_0 + \zeta\xi < \epsilon < n(e_0 + \zeta\xi)$.

Now consider the ring network. The liquidation rule

$$\zeta L_r = \left\lfloor \frac{\epsilon}{e_0 + h_0 p + \zeta\xi} \right\rfloor \zeta\xi + \left[\epsilon - \left\lfloor \frac{\epsilon}{e_0 + h_0 p + \zeta\xi} \right\rfloor (e_0 + h_0 p + \zeta\xi) - (e_0 + h_0 p) \right]^+$$

implies that there are $k(p) < n$ agents defaulting and agent $k+1$ may or may not liquidate an additional amount. Therefore, we can represent ϵ as

$$\epsilon = k(e_0 + h_0 p + \zeta\xi) + \beta(e_0 + h_0 p) + \zeta\gamma, \tag{50}$$

where $0 \leq \beta \leq 1$ and $0 \leq \gamma < \xi$ with $\gamma > 0$ only if $\beta = 1$.

Case 1.1. First, suppose that agent $k+1$ has to liquidate a positive amount. Then, the liquidation rule implies

$$\zeta L_r = \epsilon - (k+1)(e_0 + h_0 p).$$

The market clearing condition requires

$$\begin{aligned} \zeta L_r &\geq \epsilon - ne_0 \\ \Rightarrow \epsilon - (k+1)(e_0 + h_0 p) &\geq \epsilon - ne_0, \end{aligned}$$

which is trivially satisfied with strict inequality even if $p = 0$. Therefore, $L_r \geq L_c$ under this case.

Case 1.2. Now suppose that agent $k+1$ does not have to liquidate any long-term project. Then, the liquidation rule implies

$$\zeta L_r = k\zeta\xi,$$

and

$$\epsilon = (k+\beta)(e_0 + h_0 p) + k\zeta\xi.$$

From the market clearing condition,

$$\zeta L_r \geq \epsilon - ne_0 = \zeta L_c.$$

Therefore, $L_r \geq L_c$ under this case as well.

Case 2. Suppose that $cd > h_0$. Now $p^\dagger = \dfrac{d + e_0 + \zeta\xi - \epsilon}{cd - h_0}$, $\hat{\epsilon}(p) = \epsilon$ if $p \leq p^\dagger$, and $\hat{\epsilon}(p) = d + e_0 + \zeta\xi - (cd - h_0)p$ if $p > p^\dagger$.

Case 2.1. Suppose that $\dfrac{d - (n-1)e_0}{cd - h_0} \geq s\zeta$. First, consider the complete network. We guess and verify that the equilibrium price is $p > p^{\dagger}$ for the complete network. The market clearing condition is

$$\zeta L_c \geq e_0 + h_0 p + \zeta\xi + d - cdp - ne_0$$
$$\Rightarrow \zeta L_c - \zeta\xi \geq d - (n-1)e_0 - (cd - h_0)p$$
$$\Rightarrow p \geq \frac{d - (n-1)e_0 - \zeta L_c + \zeta\xi}{cd - h_0}.$$

The liquidation rule implies that $\zeta L_c = \zeta\xi$ if $p > s\zeta$, which is true only if

$$\frac{d - (n-1)e_0}{cd - h_0} \geq s\zeta.$$

Under this case, $p = \dfrac{d - (n-1)e_0}{cd - h_0}$ and $p > p^{\dagger}$ because

$$\epsilon_* = ne_0 + \zeta\xi < \epsilon$$
$$\Rightarrow d + e_0 + \zeta\xi - \epsilon < d - (n-1)e_0.$$

Hence, $L_c = \xi$ under this case.

Now consider the ring network. First, suppose that $p > p^{\dagger}$. We show that $k > 1$, so that $L_r > L_c$. This is because

$$\frac{\hat{\epsilon}(p)}{e_0 + h_0 p + \zeta\xi} = \frac{e_0 + h_0 p + \zeta\xi + d - cdp}{e_0 + h_0 p + \zeta\xi} > 1$$
$$e_0 + h_0 p + \zeta\xi + d - cdp > e_0 + h_0 p + \zeta\xi$$
$$d > cdp,$$

which is true because $cp < 1$ by $c < \underline{c}$.

Next, suppose that $p \leq p^{\dagger}$. Then, the market clearing condition implies

$$\zeta L_r \geq \epsilon - ne_0 > \zeta\xi = \zeta L_c.$$

Hence, $L_r > L_c$.

Case 2.2. Suppose that $\dfrac{d-(n-1)e_0}{cd-h_0} < s\zeta$.

Case 2.2.1. Suppose that $p > p^{\dagger}$. The market clearing condition implies

$$\begin{aligned}
\zeta L &\geq d + e_0 + \zeta\xi - (cd-h_0)p - ne_0, \\
\zeta L &= d-(n-1)e_0 + \zeta\xi - (cd-h_0)p, \qquad \text{if } p < s \\
p &= \frac{d-(n-1)e_0+\zeta\xi-\zeta L}{cd-h_0} < s\zeta,
\end{aligned}$$

by the initial assumption of Case 2.2. Hence, every agent liquidates fully, $L = n\xi$, and the equilibrium price is

$$p = \frac{d-(n-1)e_0-(n-1)\zeta\xi}{cd-h_0} < \frac{d+e_0+\zeta\xi-\epsilon}{cd-h_0} = p^{\dagger}.$$

Therefore, $p > p^{\dagger}$ is impossible.

Case 2.2.2. Now we have $p \leq p^{\dagger}$, which implies $\hat{\epsilon}(p) = \epsilon$, and this invokes the same market clearing conditions as in Case 1. Therefore, $L_r \geq L_c$ under this case as well.

Therefore, $L_r \geq L_c$ always.

Finally, consider a δ-connected network in which only agent 1 and 2 are in $\mathcal{S}$ and all other agents are in $N \backslash \mathcal{S}$. Then, agents 1 and 2 liquidate all their projects, while all other agents can fulfill their liabilities in full. Therefore,

$$\sum_{i \in N} \zeta l_i = 2\zeta\xi,$$

which is less than $\epsilon - n(e_0 + h_0 p)$ as long as $\epsilon > \epsilon_* + \zeta\xi$. Thus, the δ-connected network is strictly more stable and resilient than the complete network.

2. Now we prove the second statement. First, note that $cp < 1$ and $p < \hat{s}$ in any network structure because (34) in the proof of Lemma 9 does not hold. Also, suppose $d > d^* \equiv (n-1)(e_0 + \zeta\xi)$.

For the ring network, the same steps in Case 2 of the proof of the second statement of Proposition 2 show that the ring network is the least stable and resilient network.

Now consider the complete network. The agent under a liquidity shock defaults even on the senior debt (liquidity shock) by Lemma 7 and defaults fully on the debt obligation of $(d - cdp)/(n-1)$ to all other agents. Suppose the contrary, that $n-1$ agents do not default. Then, each agent's net wealth equation should satisfy

$$(n-2)\frac{d - cdp}{n-1} + e_0 + h_0 p + \zeta\xi - (d - cdp) \geq 0. \tag{51}$$

If any of the surviving agents liquidate a positive amount, then $p \leq s\zeta$ should hold.

Case 1. First, suppose that no surviving agents liquidate their long-term project. The market clearing condition is

$$(n-1)e_0 - d + cdp \geq h_0 p, \tag{52}$$

where $p > 0$ by the assumption that there are surviving agents. Because $d > (n-1)(e_0 + \zeta\xi)$, if $cd < h_0$, then there is no price $p > 0$ that can satisfy the market clearing condition, so $p = 0$. Then, (51) implies $(n-1)e_0 \geq d$, which is a contradiction. Now consider the case with $cd > h_0$. This inequality implies (52) becomes

$$(cd - h_0)p \geq d - (n-1)e_0, \tag{53}$$

and the left-hand side is maximized when $p = \hat{s}$, which is the case for the maximum equilibrium we are focusing on. Because $c < \underline{c} \equiv \dfrac{d-(n-1)e_0+h_0\hat{s}}{d\hat{s}}$, (53) becomes

$$d-(n-1)e_0 \leq (cd-h_0)\hat{s} < \underline{c}d\hat{s}-h_0\hat{s} = d-(n-1)e_0,$$

which is a contradiction. Therefore, $p=0$, and, again, (51) implies $(n-1)(e_0+\zeta\xi) \geq d$, which is a contradiction.

Case 2. Now some or all agents are liquidating their long-term investment projects, and the asset price should be $p>0$ if not all agents are defaulting. If $cd \leq h_0$, then the market clearing condition implies

$$0 \leq (h_0-cd)p \leq (n-1)e_0 + \sum_{j\notin\mathcal{D}} \zeta l_j - d \leq (n-1)(e_0+\zeta\xi) - d$$
$$\Rightarrow d \leq (n-1)(e_0+\zeta\xi),$$

which contradicts $d > d^*$. Hence, the asset price is $p=0$, and every agent defaults.

Consider the last case with $cd > h_0$. If $\hat{\epsilon}(p) = \min\{\epsilon, e_0+h_0p+\zeta\xi+d-cdp\} < \epsilon$, the market clearing condition can be satisfied only if the asset price is

$$p = \frac{d-(n-1)(e_0+\zeta\xi)}{cd-h_0}$$

or above. However, this price is above

$$p^\dagger = \left[\frac{d+e_0+\zeta\xi-\epsilon}{cd-h_0}\right]^+,$$

because

$$p-p^\dagger \geq \frac{\epsilon-(n-1)(e_0+\zeta\xi)}{cd-h_0} > 0,$$

by $\epsilon > \epsilon^*$ if $p^\dagger > 0$, and $p>0$ if $p^\dagger = 0$. Since $e_0+h_0p+\zeta\xi+d-cdp$ is decreasing in p with

$cd > h_0$, the equilibrium market clearing condition under $\hat{\epsilon}(p) = \epsilon$ should be

$$ne_0 + nh_0 p + n\zeta\xi - \epsilon \geq nh_0 p,$$

which cannot be satisfied again by $\epsilon > \epsilon^*$. Therefore, all agents default, and the complete network is the least resilient and least stable network.

Finally, consider a δ-connected network with $\delta < \dfrac{e_0}{(n-1)d}$ and the partition of agents sets being $(\mathcal{S}, \mathcal{S}^c)$ such that $d_{ij} \leq \delta d$ for any $i \in \mathcal{S}$ and $j \in \mathcal{S}^c$. Thus, $\sum_{j \notin S} d_{ij} \leq \delta d |\mathcal{S}^c|$ for any $i \in \mathcal{S}$. Therefore, for any $i \in \mathcal{S}$,

$$\sum_{j \in S} (d_{ij} - d_{ij} cp) \geq d - cdp - \delta(d - cdp)|\mathcal{S}^c| \geq d - cdp - e_0,$$

which implies

$$e_0 + h_0 p + \sum_{j \in S} (d_{ij} - d_{ij} cp) \geq e_0 + \sum_{j \in S} (d_{ij} - d_{ij} cp) \geq d - cdp,$$

so agents in $\mathcal{S}$ can fulfill their debt even when all agents in $\mathcal{S}^c$ do not pay any amount to agents in $\mathcal{S}$. Note that the last inequality holds for any given collateral price p. In other words, all agents in $\mathcal{S}$ remain solvent when $\omega_j = 1$ for an agent $j \in \mathcal{S}^c$ regardless of the size of the shock. Therefore, the δ-connected network is more stable and resilient than the complete or ring networks.

3. The proof of the third statement is identical to that of the case with $\zeta \to 0$. ■

D. Case with No Collateral Assets

Suppose the uniform collateral ratio is $c = 0$, and $h_0 = 0$, so there is no collateral or asset in the market. This case encompasses the main model setting of Acemoglu et al. (2015). When there are no collateral or asset holdings, the only remaining channel of contagion is the debt channel. The following result summarizes the main results of Acemoglu et al. (2015) related to the phase transition property of financial contagion depending on the size of the shock. If the shock is small, the complete network is the most resilient and stable network (robust), but if the shock is large, the complete network is the least resilient and stable network (fragile).

Proposition 8. *(Acemoglu et al., 2015) Suppose that $c = 0$ and $h_0 = 0$, so there is no collateral or asset in the market. For $\epsilon^* = ne_0$, there exists $d^* = (n-1)e_0$ such that for any $\epsilon < \epsilon^*$ and $d > d^*$, the following holds:*

1. *The complete network is the most resilient and stable.*
2. *The ring network is the least resilient and stable.*
3. *The γ-convex combination of the ring and complete networks becomes more stable and resilient as γ increases.*

Furthermore, for any $\epsilon > \epsilon^*$ and δ sufficiently small,

1. *Both the complete and ring networks are the least resilient and stable.*
2. *A δ-connected network is more resilient and stable than the complete network.*

Proof of Proposition 8. First, compute the collateral-netting network of the original network. The case of a fire-sale collapse under $p < s\zeta$ is irrelevant, as $nh_0 = 0$. Then, for this collateral-netting network, apply Propositions 4 and 6 of Acemoglu et al. (2015), and the results follow. ■

E. Numerical Analysis Additional Results

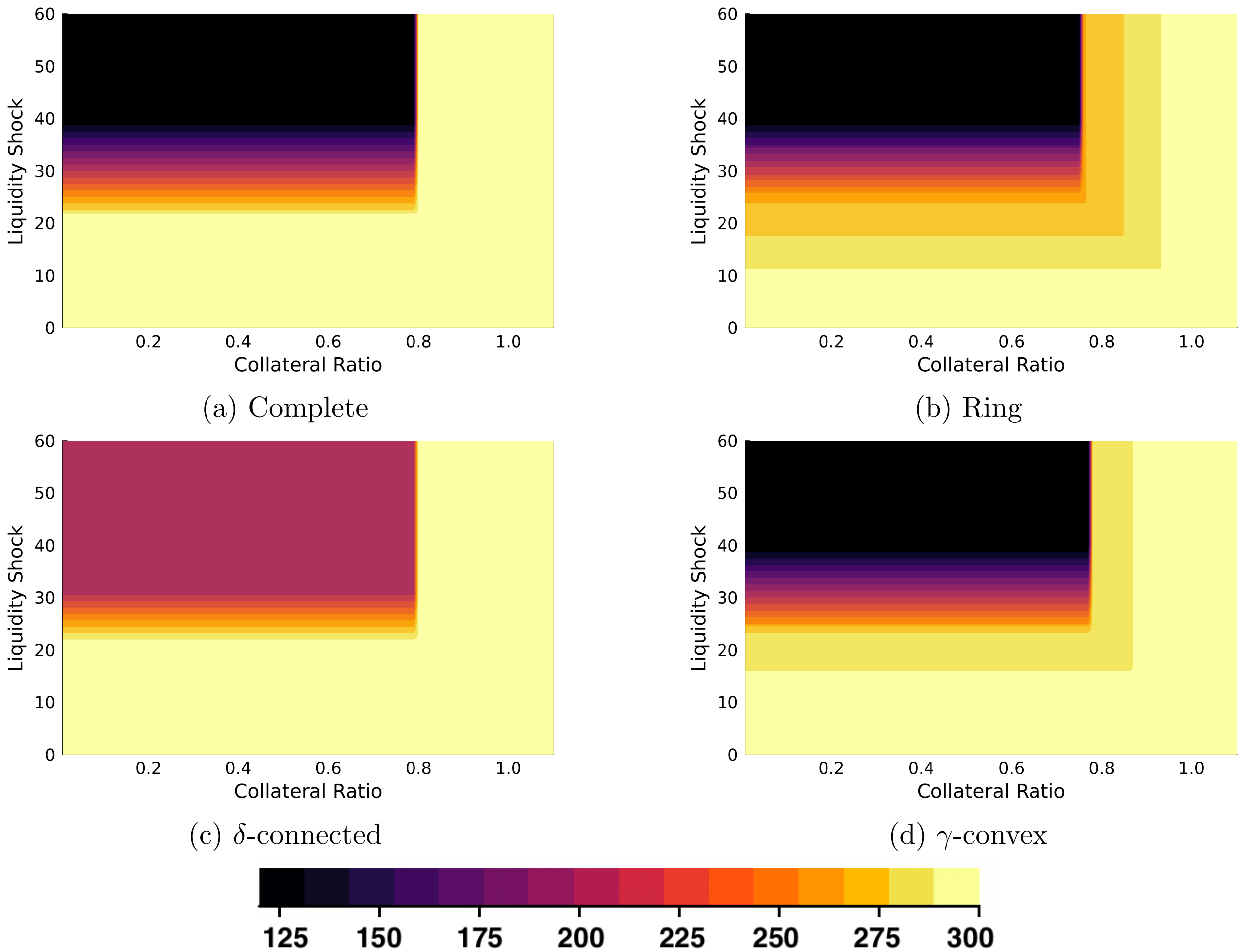


Figure 9: Social surplus under collateralized debt

Note: We evaluate 20 agents, each holding 1 cash endowment (e_0), 4 assets (h_0), and an investment project (ξ) valued at 10. Each agent owes and lends a total debt obligation (d) of $2d^* = 76$. Only 1 agent is under liquidity shock, and the liquidation efficiency (ζ) is 0.1. $\delta = 0$ for the δ-connected network, and $\gamma = 0.5$ for the γ-convex network.

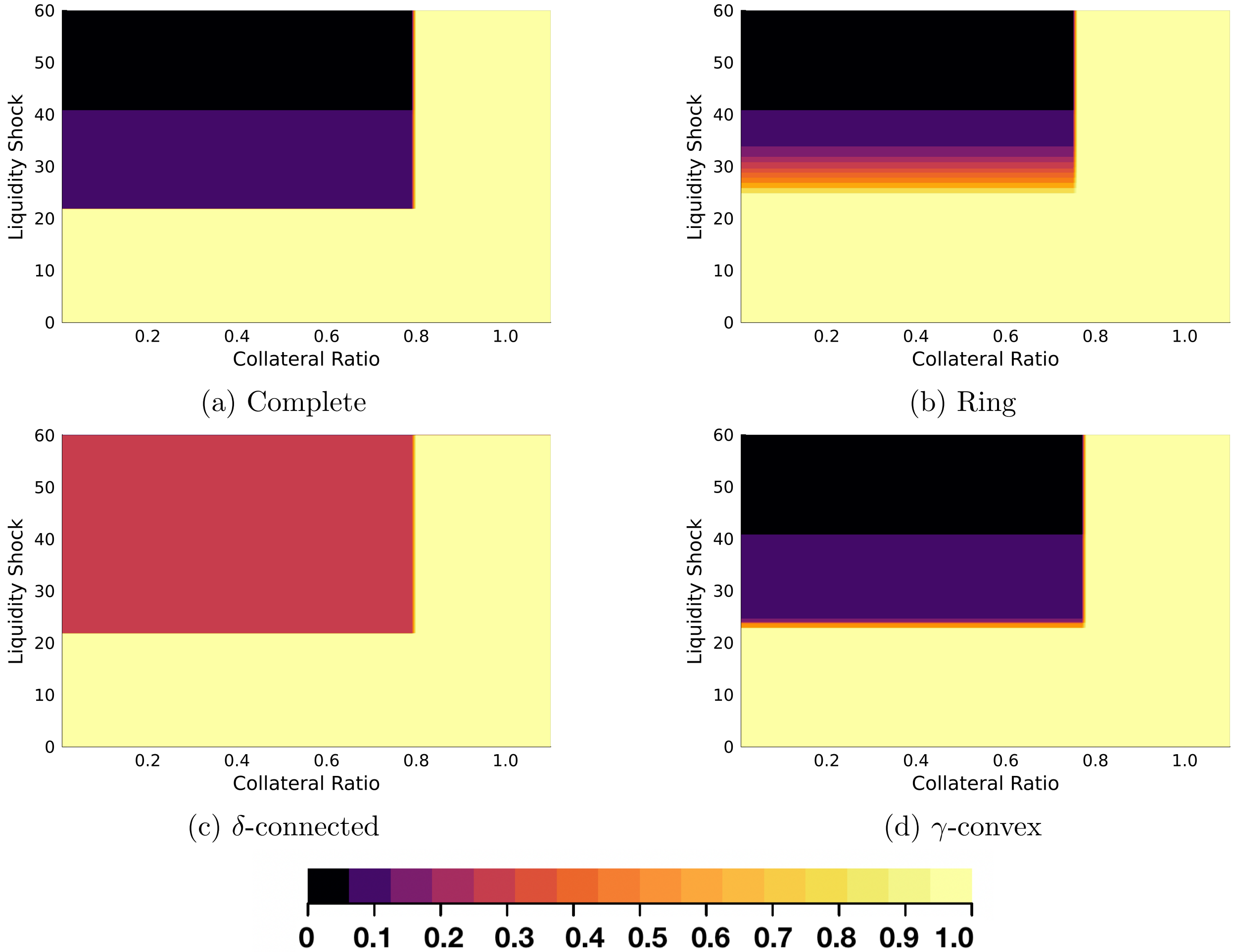
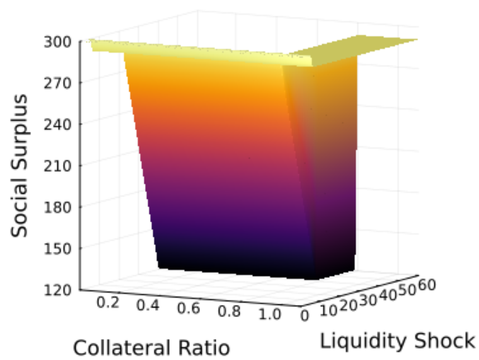
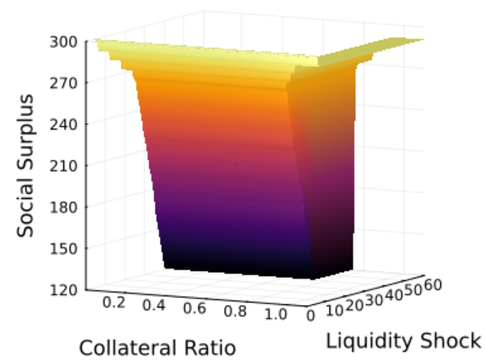


Figure 10: Price level under collateralized debt

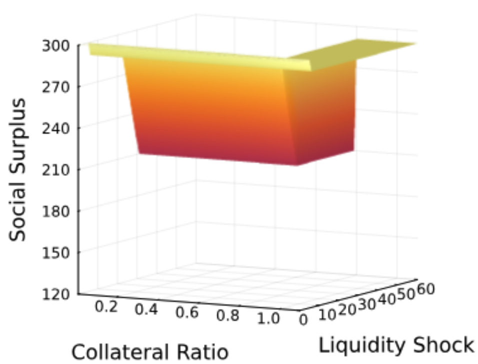
Note: We evaluate 20 agents, each holding 1 cash endowment (e_0), 4 assets (h_0), and an investment project (ξ) valued at 10. Each agent owes and lends a total debt obligation (d) of $2d^* = 76$. Only 1 agent is under liquidity shock, and the liquidation efficiency (ζ) is 0.1. $\delta = 0$ for the δ-connected network, and $\gamma = 0.5$ for the γ-convex network.



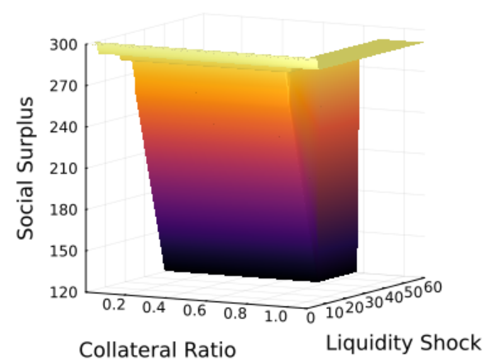
(a) Complete



(b) Ring



(c) δ-connected



(d) γ-convex

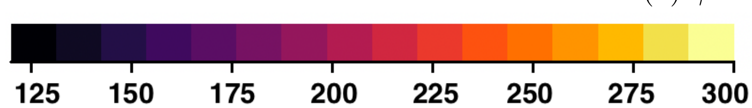
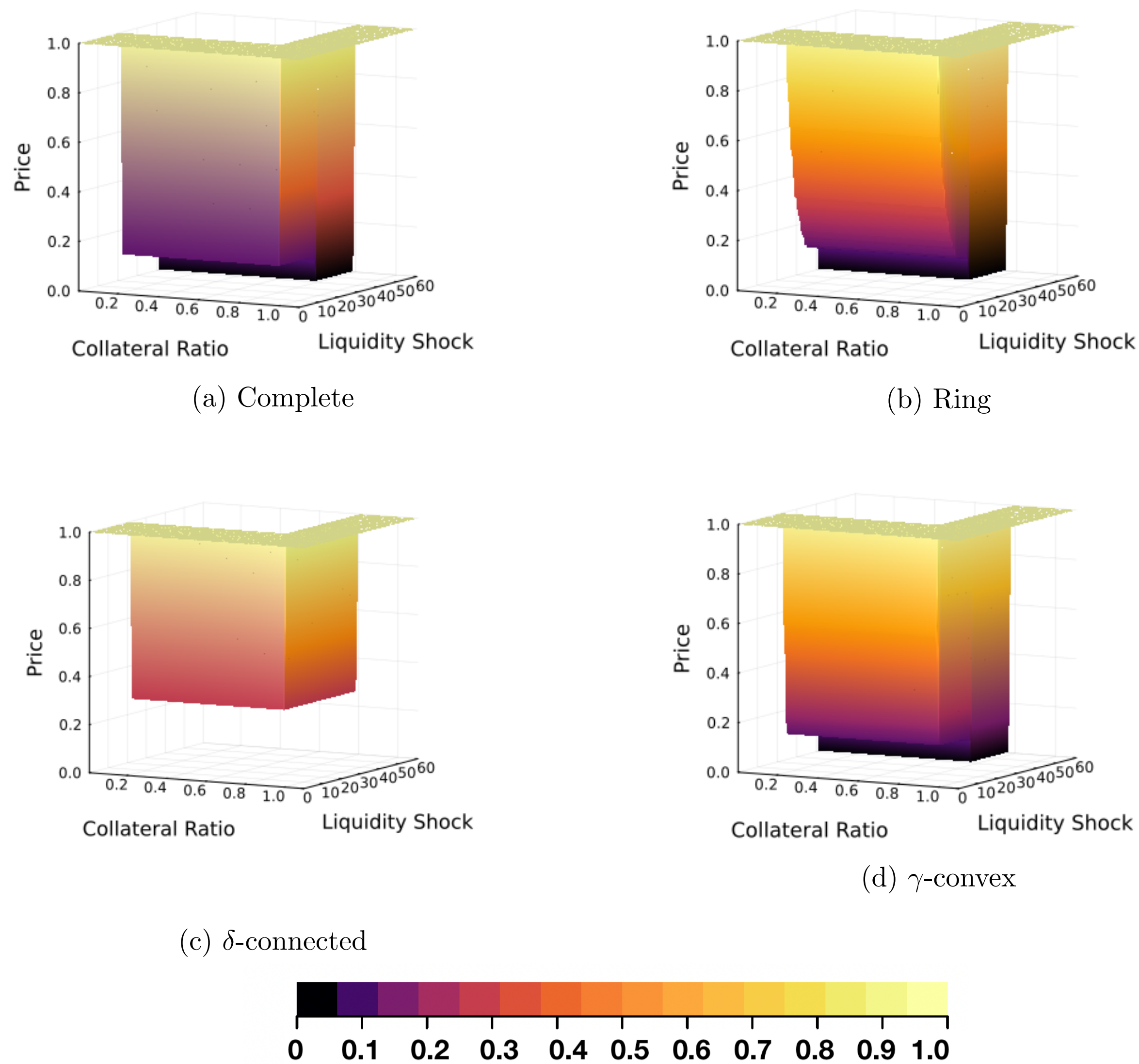


Figure 11: Social Surplus under collateralized debt (3D)

Note: We evaluate 20 agents, each holding 1 cash endowment (e_0), 4 assets (h_0), and an investment project (ξ) valued at 10. Each agent owes and lends a total debt obligation (d) of $2d^* = 76$. Only 1 agent is under liquidity shock, and the liquidation efficiency (ζ) is 0.1. $\delta = 0$ for the δ-connected network, and $\gamma = 0.5$ for the γ-convex network.



(a) Complete

(b) Ring

(c) δ-connected

(d) γ-convex

Figure 12: Price level under collateralized debt (3D)

Note: We evaluate 20 agents, each holding 1 cash endowment (e_0), 4 assets (h_0), and an investment project (ξ) valued at 10. Each agent owes and lends a total debt obligation (d) of $2d^* = 76$. Only 1 agent is under liquidity shock, and the liquidation efficiency (ζ) is 0.1. $\delta = 0$ for the δ-connected network, and $\gamma = 0.5$ for the γ-convex network.

For code that generates all numerically simulated charts, please visit https://github.com/gracechuan2/contagion/blob/main/Network%20Contagion%20Simulations_final.ipynb.