Online Appendix (for online publication only)

A. Omitted Proofs

This section contains omitted proofs of the paper.

A.1. Solving the Optimization Problem

In this subsection, I provide the detailed steps of solving the optimization problem of each agent. Recall that agent j's optimization problem is

$$\max_{\substack{e_j^1,\{c_{ij},d_{ij}\}_{i\in N},\\ a_j^1,\{c_{jk},d_{jk}\}_{k\in N}}} E_j\left[\left(e_j^1-\epsilon_j+a_j^1p_1+\textstyle\sum_{i\in N}(c_{ji}\min\{p_1,d_{ji}\}-c_{ij}\min\{p_1,d_{ij}\})-\sum_{i:m_i<0}\Psi_{ij}(C)[p_1-d_{ij}]^+\right)\frac{s}{p_1}\right]$$

$$\begin{aligned}\text{s.t. } & a_j^1+\sum_{k\in N}c_{jk}\geq\sum_{i\in N}c_{ij}, \\ & e^0=e_j^1-\sum_{i\in N}c_{ij}q_i(d_{ij})+\sum_{k\in N}c_{jk}q_j(d_{jk})+a_j^1p_0,\end{aligned} \tag{A1}$$

and μ and λ are Lagrangian multipliers for the collateral constraint and budget constraint, respectively. Denote the Lagrangian problem as $\mathcal{L}$ and the Lagrangian multipliers for nonnegativity constraints as ξ_e, $\xi_{c_{ij}}$, ξ_a, $\xi_{c_{jk}}$. The first-order conditions of the optimization problem are

$$\frac{\partial\mathcal{L}}{\partial e_j^1}=E_j\left[\frac{s}{p_1}\right]-\lambda+\xi_e=0 \tag{A2}$$

$$\frac{\partial\mathcal{L}}{\partial c_{ij}}=E_j\left[\left(-\min\{p_1,d_{ij}\}-\mathbb{1}\{i\in B(\epsilon)\}\frac{\partial\Psi_{ij}(C)}{\partial c_{ij}}[p_1-d_{ij}]^+\right)\frac{s}{p_1}\right]-\mu+\lambda q_i(d_{ij})+\xi_{c_{ij}}=0 \tag{A3}$$

$$\frac{\partial\mathcal{L}}{\partial d_{ij}}=E_j\left[(-c_{ij}+\mathbb{1}\{i\in B(\epsilon)\}\Psi_{ij}(C))\frac{s}{p_1}\middle|p_1>d_{ij}\right]\Pr_j(p_1>d_{ij})+\lambda c_{ij}q_i'(d_{ij})=0 \tag{A4}$$

$$\frac{\partial\mathcal{L}}{\partial a_j^1}=E_j[s]+\mu-\lambda p_0+\xi_a=0 \tag{A5}$$

$$\frac{\partial\mathcal{L}}{\partial c_{jk}}=E_j\left[(\min\{p_1,d_{jk}\})\frac{s}{p_1}\right]+\mu-\lambda q_j(d_{jk})+\xi_{c_{jk}}=0 \tag{A6}$$

$$\frac{\partial\mathcal{L}}{\partial d_{jk}}=E_j\left[(c_{jk})\frac{s}{p_1}\middle|p_1>d_{jk}\right]\Pr_j(p_1>d_{jk})-\lambda c_{jk}q_j'(d_{jk})=0. \tag{A7}$$

From Lemma 1, $e_j^1>0$ and $\xi_e=0$. Thus, (A2) implies

$$\lambda=E_j\left[\frac{s}{p_1}\right],$$

which is used in the proof of Lemma A4. Equations (A4) and (A7) are also used in the proof of Lemma A4.

For the borrowing and lending amounts, first consider the case that agent j's collateral constraint is not binding—that is, $\mu = 0$ and $c_{jk} > 0$ for some k. Then, (A6) becomes

$$E_j\left[\min\{p_1, d_{jk}\}\frac{s}{p_1}\right] = E_j\left[\frac{s}{p_1}\right]q_j(d_{jk}),$$

which can be rearranged as (10). This return can be interpreted as the return of lending more to k without leverage because agent j is not using this additional asset as collateral to leverage this lending.

Second, consider the case that agent j's collateral constraint is binding, and agent j is lending or purchasing with a positive amount—that is, $\mu > 0$ and $c_{ij}, c_{jk} > 0$ for some i and k. Then, (A3) and (A6) can be rearranged as

$$\begin{aligned}
\mu &= E_j\left[\left(-\min\{p_1, d_{ij}\} - \mathbb{1}\{i \in B(\epsilon)\}\frac{\partial \Psi_{ij}(C)}{\partial c_{ij}}[p_1 - d_{ij}]^+\right)\frac{s}{p_1}\right] + E_j\left[\frac{s}{p_1}\right]q_i(d_{ij}) \\
\mu &= E_j\left[\frac{s}{p_1}\right]q_j(d_{jk}) - E_j\left[(\min\{p_1, d_{jk}\})\frac{s}{p_1}\right] \\
\Rightarrow E_j\left[\frac{s}{p_1}\right] &= \frac{E_j\left[\left(\min\{p_1, d_{jk}\} - \min\{p_1, d_{ij}\} - \mathbb{1}\{i \in B(\epsilon)\}\frac{\partial \Psi_{ij}(C)}{\partial c_{ij}}[p_1 - d_{ij}]^+\right)\frac{s}{p_1}\right]}{q_j(d_{jk}) - q_i(d_{ij})},
\end{aligned}$$

which is the return on lending with leverage, because j is immediately using the additional asset from lending to k as collateral to borrow from i. Also, the previous equations are the derivation of (11).

Third, for an agent with $a_j^1 > 0$, (A5) implies

$$E_j\left[\frac{s}{p_0}\right] + \frac{\mu}{p_0} = E_j\left[\frac{s}{p_1}\right],$$

which is used in the proof of Lemma A5. If agent j leverages this asset purchase, then (A3) and (A5) imply

$$\begin{aligned}
\mu &= E_j\left[\left(-\min\{p_1, d_{ij}\} - \mathbb{1}\{i \in B(\epsilon)\}\frac{\partial \Psi_{ij}(C)}{\partial c_{ij}}[p_1 - d_{ij}]^+\right)\frac{s}{p_1}\right] + E_j\left[\frac{s}{p_1}\right]q_i(d_{ij}) \\
\mu &= E_j\left[\frac{s}{p_1}\right]p_0 - E_j[s] \\
\Rightarrow E_j\left[\frac{s}{p_1}\right] &= \frac{E_j\left[\left(p_1 - \min\{p_1, d_{ij}\} - \mathbb{1}\{i \in B(\epsilon)\}\frac{\partial \Psi_{ij}(C)}{\partial c_{ij}}[p_1 - d_{ij}]^+\right)\frac{s}{p_1}\right]}{p_0 - q_i(d_{ij})},
\end{aligned}$$

which is how (A15) is derived.

A.2. Preliminary Lemmas

Lemma A1. *For any given collateralized debt network under intermediation order, the effective demand $[m_j(p)]^+$ is increasing in p for any $j \in N$.*

Proof of Lemma A1. It is enough to show that $m_j(p)$, which is

$$e_j^1 - \epsilon_j + a_j^1 + \sum_{k \in N} c_{jk} \min\{p, d_{jk}\} - \sum_{i \in N} c_{ij} \min\{p, d_{ij}\} - \sum_{i \in B(\epsilon)} \Psi_{ij}(C)[p - d_{ij}]^+,$$

is increasing in p. Because $\min\{d_{ij}, p\} \leq p$, both $\min\{p, d_{ij}\}$ and $\min\{d_{jk}, p\}$ are increasing in p. For any value of promise $\hat{d}$,

$$\sum_{\substack{i \in N \\ d_{ij} \geq \hat{d}}} c_{ij} \min\{d_{ij}, p\} \leq \sum_{\substack{k \in N \\ d_{jk} \geq \hat{d}}} c_{jk} \min\{d_{jk}, p\} + a_j^1$$

by intermediation order. Therefore, the sum of the payments from other agents will always exceed the sum of payments that j has to pay to others.[18] Also, by assumption 1, $\Psi_{ij}(C) \leq c_{ij}$, the total sum of coefficients for p will always be non-negative. For fixed $B(\epsilon|s)$, each $m_j(p)$ is increasing in p. Therefore, for any $p' < p$, $B(\epsilon|s, p) \subseteq B(\epsilon|s, p')$ and the indicator function for the bankruptcy cost is decreasing in p as well. ■

The following lemma shows that whenever leveraging is profitable for a certain investment, the same leverage makes another investment more profitable than not leveraging.

Lemma A2. *Suppose $\frac{a-p}{b-q} = \pi = \frac{c-p}{d-q}$, $\frac{e}{f} \geq \pi$ and $\frac{a}{b} < \frac{a-p}{b-q}$ for $a, b, c, d, e, f, p, q, \pi > 0$. Then, $\frac{c}{d} < \frac{c-p}{d-q}$ and $\frac{e}{f} < \frac{e-p}{f-q}$.*

Proof of Lemma A2. Since $\frac{a-p}{b-q} = \pi$, $a - p = b\pi - q\pi$. By $\frac{a}{b} < \frac{a-p}{b-q}$, I obtain $a < b\pi$. By combining the previous equation and inequality, I have $p < q\pi$. Now suppose that $\frac{c}{d} \geq \frac{c-p}{d-q}$. Then, $\frac{c-p}{d-q} = \pi$ implies $c \geq d\pi$. Combining this with $p < q\pi$, I get $c < d\pi$, which is a contradiction. Therefore, $\frac{c}{d} < \frac{c-p}{d-q}$. Similarly, suppose $\frac{e}{f} \geq \frac{e-p}{f-q}$. Then, I have $\frac{e}{f} \leq \frac{p}{q} < \pi$, which contradicts the assumption $\frac{e}{f} \geq \pi$. Thus, $\frac{e}{f} < \frac{e-p}{f-q}$. ■

[18] This is, in fact, the reason why there are collateral constraints. It guarantees the agent to have a non-negative amount of cash from all the payments netted out so that they can actually pay the debt.

A.3. Properties of Payment Equilibria

Proof of Proposition 1. If $p = s$, then I automatically have an equilibrium that satisfies inequality (4) or otherwise p cannot be s. Now suppose $p < s$. The equilibrium equation can be represented as

$$(m, p) = \left([m_j(p)]_{j \in N}, \frac{\sum_{i \in N} [m_i(p)]^+}{\sum_{j \in N} a_j^1} \right) \equiv \mathcal{M}[(m, p)].$$

Consider an ordering $\succeq$ such that $(m, p) \succeq (m', p')$ when $m \geq m'$ and $p \geq p'$. Then an infimum under $\succeq$ can always be defined for any subset of $\mathbb{R}^{n+1}$. By the assumption, $(m(s), s) \geq \mathcal{M}[(m(s), s)]$. Since the denominator of the price equation is constant and $a_i^2(p)$ and $[m_i(p)]^+$ for any $i \in N$ are increasing in p by Lemma A1, the function $\mathcal{M}$ is an order-preserving function. Then, by Knaster-Tarski's fixed point theorem, there exists a fixed point (m^*, p^*), and the set of such fixed points that satisfy the equilibrium condition has a maximal point.

If equation (3) is true when $p = 0$, then I already have a fixed point with $p \leq s$. Now suppose that the maximal fixed point price $\bar{p}$ is greater than s, and I will show that either there exists a price $0 < p \leq s$ that is also a fixed point or $p = s$ satisfies equilibrium condition (4). If equation (3) is not true when $p = 0$, then that implies at least some $m_j(0)$ is positive for $j \in N$. Therefore, $\frac{\sum_{i \in N} [m_i(p)]^+}{\sum_{j \in N} a_j^1} > 0$ for any $p > 0$. This implies that as p increases, the difference between the p and $\frac{\sum_{i \in N} [m_i(p)]^+}{\sum_{j \in N} a_j^1}$ will be eventually closed out at $\bar{p}$ by intermediate value theorem. Therefore, the two functions either meet for some $p \leq s$, or the gap between them does not close out even when $p = s$ so equation (4) holds. ■

Proof of Proposition 2. If agent j is bankrupt under the original equilibrium, then the statements hold with equality. Suppose that agent j is not bankrupt under ϵ—that is, $j \notin B(\epsilon|s)$. Because of the increase in the liquidity shock, m_j^* is decreasing. Also, if the original equilibrium price was liquidity constrained, $p^* = \pi(p^*) < s$, then the new equilibrium price decreases by (3). This could further decrease m_i^* for $i \neq j$ by Lemma A1. It could also trigger additional bankruptcy of i or j and lender default loss $\beta_i(p)$ that will decrease wealth of i's borrowers $\mathcal{V}_i$ further, and the price of the asset will decrease even further by (7). The same argument goes through with the increase in lender default losses Ψ_{jk} and the decrease in cash holding e_j^1 for any $j, k \in N$. ■

A.4. Results on Network Equilibrium

Proof of Lemma 1. For each agent $j \in N$, the maximum cash the agent can hold for $t = 1$ is by saving all the cash while not lending any cash because borrowing requires collateral and no arbitrage condition will prevent anyone from making positive cash from borrowing. The price of the asset at $t = 0$ cannot exceed the most optimistic agent's fair value since there is always a possibility of liquidity constrained underpricing in $t = 1$. Thus, $e^0 + As_1$ is always the upper bound of the maximum amount of cash each agent can hold while holding all the asset endowments and not borrowing or lending at all. Since G is differentiable with full support of $[0, \bar{\epsilon}]$, any agent can go bankrupt regardless of how much cash they hold in $t = 0$ because $G([e^0 + As_1, \bar{\epsilon}])$ is positive. Now suppose that agent j has zero cash holdings, $e^1_j = 0$. Agent j's nominal wealth becomes zero if $p_1 = 0$. By equation (7), this implies that if every other agent goes bankrupt because of liquidity shocks while agent j is not, which happens with positive probability, the price of the asset becomes zero while agent j is not bankrupt. Marginal utility of cash in such a state becomes $\lim_{p_1 \to 0} \frac{s_j}{p_1}$, which is infinity. Hence, expected marginal utility of holding cash in $t = 0$ becomes infinity as well and agent j would like to hold a positive amount of cash for any $j \in N$. If $e^1_j > 0$, then the only state with infinite marginal utility of cash is when $\epsilon_j = e^1_j$, which happens with zero probability by differentiability of G. Thus, in an equilibrium, $e^1_j > 0$ for any $j \in N$. ■

The proof of Theorem 1 is based on the following three lemmas. First, Lemma A3 establishes that interest rate of the same contract increases over the lender's optimism—that is, optimistic agents demand higher interest rates. Second, Lemma A4 implies that contracts traded in positive amount should have maximum leverage by promising the fundamental value of the asset in the lender's perspective. Third, Lemma A5 pins down the natural buyers of each contract and the asset. Therefore, the three lemmas combined construct the pattern of intermediation and positively traded contracts in a network equilibrium.

Lemma A3 (Cash Return Ordering)**.** *For any two agents in a network equilibrium, the cash return from the more optimistic agent is always greater than the cash return from the less optimistic agent—that is,* $E_j\left[\frac{s_j}{p_1}\right] > E_k\left[\frac{s_k}{p_1}\right]$ *for any* $j < k$ *and* $j, k \in N$.

Proof of Lemma A3. The proof is done by contradiction. Suppose that $E_j\left[\frac{s_j}{p_1}\right] \leq E_k\left[\frac{s_k}{p_1}\right]$ for $j < k$. If both j and k are simply holding cash exclusively, then they have the same cash holdings and it is trivially $E_j\left[\frac{s_j}{p_1}\right] > E_k\left[\frac{s_k}{p_1}\right]$. Therefore, at least agent k should be investing in something other than cash. Suppose that agent k is borrowing from i and lending to l. Then, agent k's marginal return from this intermediation is

$$\frac{E_k\left[\min\left\{s_k, d'\frac{s_k}{p_1}\right\} - \min\left\{s_k, d\frac{s_k}{p_1}\right\} - \frac{\partial\Psi_{ik}(C)}{\partial c_{ik}}\left[s_k - d\frac{s_k}{p_1}\right]\mathbb{1}\left\{i \in B(\epsilon)\right\}\right]}{q_k(d') - q_i(d)}$$

$$= \frac{s_k E_k\left[\min\left\{1, \frac{d'}{p_1}\right\} - \min\left\{1, \frac{d}{p_1}\right\} - \frac{\partial\Psi_{ik}(C)}{\partial c_{ik}}\left[1 - \frac{d}{p_1}\right]\mathbb{1}\left\{i \in B(\epsilon)\right\}\right]}{q_k(d') - q_i(d)} = E_k\left[\frac{s_k}{p_1}\right].$$

The last equality holds because the return should be equal to the return from holding cash because of positive cash holding by Lemma 1. Now consider an agent j who deviates from the equilibrium portfolio decision. Agent j can mimic the investment portfolio of agent k and obtain the return of

$$\frac{s_j E_j\left[\min\left\{1, \frac{d'}{p_1}\right\} - \min\left\{1, \frac{d}{p_1}\right\} - \frac{\partial\Psi_{ik}(C)}{\partial c_{ik}}\left[1 - \frac{d}{p_1}\right]\mathbb{1}\left\{i \in B(\epsilon)\right\}\right]}{q_k(d') - q_i(d)} \leq E_j\left[\frac{s_j}{p_1}\right],$$

with the last inequality coming from the optimality of agent j's original portfolio decision. In other words, agent j would have already done the intermediation more if it exceeded the return from agent j's cash holdings (which is again positive by Lemma 1). If agent j is mimicking k's portfolio exactly, the two agents will have the same cash holdings and also the same counterparty risks (or even less if j was the lender). Then, inequalities

$$\begin{aligned}
&E_j\left[\min\left\{1, \frac{d'}{p_1}\right\} - \min\left\{1, \frac{d}{p_1}\right\} - \frac{\partial\Psi_{ik}(C)}{\partial c_{ik}}\left[1 - \frac{d}{p_1}\right]\mathbb{1}\left\{i \in B(\epsilon)\right\}\right] \\
\geq &E_k\left[\min\left\{1, \frac{d'}{p_1}\right\} - \min\left\{1, \frac{d}{p_1}\right\} - \frac{\partial\Psi_{ik}(C)}{\partial c_{ik}}\left[1 - \frac{d}{p_1}\right]\mathbb{1}\left\{i \in B(\epsilon)\right\}\right],
\end{aligned}$$

and $s_j > s_k$ imply

$$\begin{aligned}
E_j\left[\frac{s_j}{p_1}\right] \geq &\frac{s_j E_j\left[\min\left\{1, \frac{d'}{p_1}\right\} - \min\left\{1, \frac{d}{p_1}\right\} - \frac{\partial\Psi_{ik}(C)}{\partial c_{ik}}\left[1 - \frac{d}{p_1}\right]\mathbb{1}\left\{i \in B(\epsilon)\right\}\right]}{q_k(d') - q_i(d)} \\
> &\frac{s_k E_k\left[\min\left\{1, \frac{d'}{p_1}\right\} - \min\left\{1, \frac{d}{p_1}\right\} - \frac{\partial\Psi_{ik}(C)}{\partial c_{ik}}\left[1 - \frac{d}{p_1}\right]\mathbb{1}\left\{i \in B(\epsilon)\right\}\right]}{q_k(d') - q_i(d)} = E_k\left[\frac{s_k}{p_1}\right],
\end{aligned}$$

that is, $E_j\left[\frac{s_j}{p_1}\right] > E_k\left[\frac{s_k}{p_1}\right]$, which contradicts the initial assumption $E_j\left[\frac{s_j}{p_1}\right] \leq E_k\left[\frac{s_k}{p_1}\right]$. The same method could be applied to any other possible investment strategy of agent k . Therefore, $E_j\left[\frac{s_j}{p_1}\right] > E_k\left[\frac{s_k}{p_1}\right]$ holds for any equilibrium. ■

Lemma A4 (Maximum Leverage)**.** *Suppose that agent j buys an asset or a contract and borrows from agent i in a network equilibrium. Then, agent j maximizes the contract leverage by borrowing the maximum amount of cash j can borrow from agent i, which is s_i.*

Proof of Lemma A4.

From the return equation (11), I immediately get $d' > d$, and $q_j(d') > q_i(d)$ should hold for agent j's decision optimality and no arbitrage.[19] Similarly, from the positive cash holdings by Lemma 1 and optimality, and by Leibniz integral rule,

$$q_i'(d) = \frac{E_i\left[\frac{1}{p_1}\Big|p_1 > d\right]\Pr_i(p_1 > d)}{E_i\left[\frac{1}{p_1}\right]},$$

which is zero for any $d > s_i$. The partial derivative (left derivative if $d = s_i$) for agent j's decision on the contract promise choice d to agent i is

$$\begin{aligned}
&s_j E_j\left[-\frac{c_{ij}}{p_1} + \Psi_{ij}(C)\frac{1}{p_1}\mathbb{1}\left\{i \in B(\epsilon)\right\}\Big|p_1 > d\right]\Pr_j(p_1 > d) + \lambda c_{ij} q_i'(d)\\
=&s_j E_j\left[-\frac{c_{ij}}{p_1}\Big|p_1 > d\right]\Pr_j(p_1 > d) + s_j E_j\left[\Psi_{ij}(C)\frac{1}{p_1}\mathbb{1}\left\{i \in B(\epsilon)\right\}\Big|p_1 > d\right]\Pr_j(p_1 > d)\\
&+ s_j E_j\left[\frac{1}{p_1}\right]c_{ij}\frac{E_i\left[\frac{1}{p_1}\Big|p_1 > d\right]\Pr_i(p_1 > d)}{E_i\left[\frac{1}{p_1}\right]},
\end{aligned}$$

where λ is the Lagrangian multiplier for the budget constraint. From Lemma 1 and the first-order condition with respect to e_j^1, we have $\lambda = s_j E_j[1/p_1]$. First, if $d > s_i$, then the last term is zero. Since $c_{ij} > \Psi_{ij}(C)$, the first-order derivative is negative for any $d > s_i$. Now consider $d \leq s_i$. I show that the above first-order derivative is positive, even if the counterparty risk is zero, by showing the following inequality for any $d \leq s_i$,

$$\frac{E_j\left[\frac{1}{p_1}\Big|p_1 > d\right]\Pr_j(p_1 > d)}{E_j\left[\frac{1}{p_1}\right]} < \frac{E_i\left[\frac{1}{p_1}\Big|p_1 > d\right]\Pr_i(p_1 > d)}{E_i\left[\frac{1}{p_1}\right]}. \tag{A8}$$

Suppose that the above inequality does not hold—that is,

$$\frac{E_j\left[\frac{1}{p_1}\Big|p_1 > d\right]\Pr_j(p_1 > d)}{E_j\left[\frac{1}{p_1}\right]} \geq \frac{E_i\left[\frac{1}{p_1}\Big|p_1 > d\right]\Pr_i(p_1 > d)}{E_i\left[\frac{1}{p_1}\right]}. \tag{A9}$$

[19] No arbitrage prevents the case of $d' < d$ and $q_j(d') < q_i(d)$.

From Lemma A3, the cash return of j should exceed that of i as

$$\begin{aligned} E_j\left[\frac{s_j}{p_1}\right] =& s_j\left(E_j\left[\frac{1}{p_1}\middle|p_1 > d\right]\Pr_j(p_1 > d) + E_j\left[\frac{1}{p_1}\middle|p_1 \le d\right]\Pr_j(p_1 \le d)\right) \\ >& s_i\left(E_i\left[\frac{1}{p_1}\middle|p_1 > d\right]\Pr_i(p_1 > d) + E_i\left[\frac{1}{p_1}\middle|p_1 \le d\right]\Pr_i(p_1 \le d)\right) = E_i\left[\frac{s_i}{p_1}\right], \end{aligned}$$

which can be rearranged as

$$\begin{aligned} &\frac{1}{s_j\left(E_j\left[\frac{1}{p_1}\middle|p_1 > d\right]\Pr_j(p_1 > d) + E_j\left[\frac{1}{p_1}\middle|p_1 \le d\right]\Pr_j(p_1 \le d)\right)} \\ <&\frac{1}{s_i\left(E_i\left[\frac{1}{p_1}\middle|p_1 > d\right]\Pr_i(p_1 > d) + E_i\left[\frac{1}{p_1}\middle|p_1 \le d\right]\Pr_i(p_1 \le d)\right)}. \end{aligned} \tag{A10}$$

By the assumption (A9),

$$\begin{aligned} &\frac{s_jE_j\left[\frac{1}{p_1}\middle|p_1 > d\right]\Pr_j(p_1 > d)}{s_j\left(E_j\left[\frac{1}{p_1}\middle|p_1 > d\right]\Pr_j(p_1 > d) + E_j\left[\frac{1}{p_1}\middle|p_1 \le d\right]\Pr_j(p_1 \le d)\right)} \\ \ge&\frac{s_iE_i\left[\frac{1}{p_1}\middle|p_1 > d\right]\Pr_i(p_1 > d)}{s_i\left(E_i\left[\frac{1}{p_1}\middle|p_1 > d\right]\Pr_i(p_1 > d) + E_i\left[\frac{1}{p_1}\middle|p_1 \le d\right]\Pr_i(p_1 \le d)\right)}, \end{aligned}$$

which implies that

$$\frac{s_jE_j\left[\frac{1}{p_1}\middle|p_1 > d\right]\Pr_j(p_1 > d)}{s_iE_i\left[\frac{1}{p_1}\middle|p_1 > d\right]\Pr_i(p_1 > d)} > \frac{s_jE_j\left[\frac{1}{p_1}\right]}{s_iE_i\left[\frac{1}{p_1}\right]}.$$

Since the upper bound for price under agent j's perspective, s_j, is higher than that under agent i's perspective, s_i, the previous inequality holds only if $\Pr_j(p_1 > d)$ is much larger than $\Pr_i(p_1 > d)$. However, then $\Pr_i(p_1 \le d) > \Pr_j(p_1 \le d)$ and $1/p_1$ is larger when $p_1 \le d$ than $1/p_1$ when $p_1 > d$. Therefore,

$$\frac{s_jE_j\left[\frac{1}{p_1}\right]}{s_iE_i\left[\frac{1}{p_1}\right]} < 1,$$

which violates (A10). Therefore, the assumption (A9) is false, and (A8) holds, which implies the first-order derivative (left derivative) is positive for any $d \le s_i$. Hence, agent j promises s_i and maximizes agent j's leverage. ■

Lemma A5 (Natural Buyers)**.** *In a network equilibrium, the following statements are true:*

1. *The most optimists, agent 1, buys some or all of the asset, $a_{1,1} > 0$, and $p_0 = q_1(s_1)$.*
2. *For any agent $i < n$, i borrows from agent $i+1$ with positive amount, $c_{i+1,i} > 0$.*
3. *For any agent $i < n-1$, if $c_{ji} > 0$, and i's perceived marginal counterparty risks of j and k are the same for $i < j < k$, then i marginally prefers borrowing more from agent j to borrowing more from agent k.*

Proof of Lemma A5. First, consider the option of purchasing the asset without leverage. Suppose agent $j > 1$ is buying the asset while agent 1 is not buying. Return from the asset purchase for agent j is s_j/p_0. By Lemma 1, agent j should equate the returns from cash and asset as

$$\frac{s_j}{p_0} = E_j\left[\frac{s_j}{p_1}\right].$$

But then, $\frac{s_j}{p_0} < \frac{s_1}{p_0} < E_1\left[\frac{s_1}{p_1}\right]$ because agent 1 does not purchase the asset. Hence,

$$s_j E_j\left[\frac{1}{p_1}\right] = \frac{s_j}{p_0} < \frac{s_1}{p_0} < s_1 E_1\left[\frac{1}{p_1}\right] < s_1 E_j\left[\frac{1}{p_1}\right] = \frac{s_1}{p_0},$$

where the last inequality comes from the fact that agent j has less cash and is more likely to experience severe under pricing as well as a lower upper bound for price p_1, and the above inequality leads to a contradiction. This implies agent j would rather sell the asset to agent 1 and both make profitable trades. The same inference can be done with levered purchases, as both agents can do the same borrowing from the same set of lenders and simply change the price as the down payment such as $p_0 - q(s_i)$.

The second statement holds with the similar argument in the proof of the first statement. The problem for agent i becomes isomorphic to agent 1's optimization by substituting the asset with the promise of s_i by agent $i-1$, which is coming from Lemma A4. Then, I can apply the same logic as in the first statement. Agent i can always mimic an agent who is more pessimistic and purchasing the contract, and increase payoff for the given price.

For the third statement, denote the implied *expected lender default from agent i under j's belief* as $\omega_{ij}(d;C) \equiv E_j\left[[p_1 - d]^+ \mathbb{1}[i \in B(\epsilon|s)]\right]$. The *counterparty risk of borrowing from agent i for j* is $\Psi_{ij}(C)\omega_{ij}(d;C)$. Then, the marginal returns from a leveraged position is

$$R_i^j \equiv \frac{s_i}{q_i(s_i) - q_j(s_j)} E_i\left[\min\left\{1, \frac{s_i}{p_1}\right\} - \min\left\{1, \frac{s_j}{p_1}\right\}\right] + \Psi'_{ji}(C)\omega_{ji}(s_j;C)$$

for agents $i < j$. First start with agents as $i = 1,\ j = 2,\ k = 3$. Suppose that agent 1 has the same marginal counterparty risks for agent 2 and 3. By the first and second statements, agent 1 buys the asset and agent 2 lends to agent 1 that promises s_2. By the first and second statements, buying the asset and borrowing from agent 2 should be one of the optimal choices for agent 1. By Lemma 1, the return from this decision should be equal to the cash return for agent 1—that is, $R_1^2 = E_1\left[\frac{s_1}{p_1}\right]$.

Now suppose that the third statement is not true—that is, $R_1^2 \leq R_1^3$. If $R_1^2 < R_1^3$, then agent 1 does not borrow from agent 2, which contradicts the second statement. Therefore, the only case left to check is $R_1^2 = R_1^3$. Then, both returns should equal the cash return

$$\frac{s_1 E_1\left[\min\left\{1, \frac{s_2}{p_1}\right\}\right]}{q_2(s_2)} = \frac{s_1 E_1\left[\min\left\{1, \frac{s_3}{p_1}\right\}\right]}{q_3(s_3)}.$$

By the previous two statements of the lemma, agent 1's leveraged purchase by borrowing from agent 2 should be profitable and the difference in expected payment of s_3 to agent 3 between agent 1 and 2 cannot exceed their difference in beliefs. Thus,

$$\frac{s_2 E_2\left[\min\left\{1, \frac{s_2}{p_1}\right\}\right]}{q_2(s_2)} < \frac{s_1 E_1\left[\min\left\{1, \frac{s_2}{p_1}\right\}\right]}{q_2(s_2)} = \frac{s_1 E_1\left[\min\left\{1, \frac{s_3}{p_1}\right\}\right]}{q_3(s_3)} < \frac{s_2 E_2\left[\min\left\{1, \frac{s_3}{p_1}\right\}\right]}{q_3(s_3)}.$$

But, then $\frac{s_2 E_2\left[\min\left\{1, \frac{s_2}{p_1}\right\}\right]}{q_2(s_2)} < \frac{s_2 E_2\left[\min\left\{1, \frac{s_3}{p_1}\right\}\right]}{q_3(s_3)}$ implies that agent 2 does not want to borrow from agent 3, which contradicts the second statement. Therefore, $R_1^2 > R_1^3$. In fact, the above arguments hold for any three consecutive agents $i,\ i+1,\ i+2$ for $i < n-1$.

Now I extend the case to consider any arbitrary agents $i < j < k$ with $i < n-1$. Suppose that $j = i+1$ and $k > i+1$ and $R_i^j \leq R_i^k$. Again, by the same argument, the only possible case left is $R_i^j = R_i^k$. Then, by the similar process for the previous case

$$\begin{aligned}\frac{s_j E_j\left[\min\left\{1, \frac{s_k}{p_1}\right\}\right]}{q_k(s_k)} &\leq \frac{s_j E_j\left[\min\left\{1, \frac{s_{j+1}}{p_1}\right\}\right]}{q_{j+1}(s_{j+1})} < \frac{s_i E_i\left[\min\left\{1, \frac{s_j}{p_1}\right\}\right]}{q_j(s_j)} \\ &= \frac{s_i E_i\left[\min\left\{1, \frac{s_k}{p_1}\right\}\right]}{q_k(s_k)} < \frac{s_j E_j\left[\min\left\{1, \frac{s_k}{p_1}\right\}\right]}{q_k(s_k)},\end{aligned}$$

which is again a contradiction.

Finally, I can apply these results to show that $R_i^j > R_i^k$ is true for any arbitrary $i < j < k$.

This is because

$$
\begin{aligned}
&\frac{s_j E_j\left[\min\left\{1,\dfrac{s_k}{p_1}\right\}\right]}{q_k(s_k)} \leq \frac{s_j E_j\left[\min\left\{1,\dfrac{s_{j+1}}{p_1}\right\}\right]}{q_{j+1}(s_{j+1})} < \frac{s_{j-1} E_{j-1}\left[\min\left\{1,\dfrac{s_j}{p_1}\right\}\right]}{q_j(s_j)} < \cdots \\
&< \frac{s_{i+1} E_{i+1}\left[\min\left\{1,\dfrac{s_{i+2}}{p_1}\right\}\right]}{q_{i+2}(s_{i+2})} < \frac{s_i E_i\left[\min\left\{1,\dfrac{s_{i+1}}{p_1}\right\}\right]}{q_{i+1}(s_{i+1})} = \frac{s_i E_i\left[\min\left\{1,\dfrac{s_k}{p_1}\right\}\right]}{q_k(s_k)} \\
&< \frac{s_j E_j\left[\min\left\{1,\dfrac{s_k}{p_1}\right\}\right]}{q_k(s_k)},
\end{aligned}
$$

which is coming from the previous arguments and again generates a contradiction. Therefore, $R_i^j > R_i^k$ and agent $i < n-1$ prefers to borrow more from j over k for any $i < j < k$. ■

Proof of Theorem 1. By Lemma A3, no agent will borrow from a more optimistic agent. Then, by the collateral constraint,

$$\sum_{\substack{i \in N \\ d_{ij} \geq \hat{d}}} c_{ij} \leq a_j^1 + \sum_{\substack{k \in N \\ d_{jk} \geq \hat{d}}} c_{jk}$$

should hold for any debt level $\hat{d} \in \mathbb{R}^+$ and for any $j \in N$. Therefore, the equilibrium collateralized debt network is under intermediation order. By Lemma A4, agents' optimal contract choice is promising the fundamental value of the asset in lender's perspective, and the optimization problem becomes choosing weights of their collateral exposure to different lenders. Therefore, only the kink points—$s_1, s_2, \ldots, s_n$—will be traded in any equilibrium. By Lemma A5, agent j is borrowing a positive amount from $j+1$ for any $j < n$. Hence, any agent who is willing to borrow from agent j faces the contract price of

$$q_j(d) = q_{j+1}(s_{j+1}) + \frac{E_j\left[\min\left\{1,\dfrac{d}{p_1}\right\} - \min\left\{1,\dfrac{s_{j+1}}{p_1}\right\} - \dfrac{\partial \Psi_{j+1,j}(C)}{\partial c_{j+1,j}}\left[1-\dfrac{s_{j+1}}{p_1}\right]^+ \mathbb{1}\{j+1 \in B(\epsilon)\}\right]}{E_j\left[\dfrac{1}{p_1}\right]}.$$

Note that the above contract price is the no-arbitrage (or break-even) price for lender j. Hypothetically, if someone wants to borrow from agent 1 promising s_1 in $t=1$, then agent 1 is willing to lend $q_1(s_1)$ to the borrower in $t=0$. From agent 1's perspective, this contract is equivalent to the payoff from purchasing the asset at price of $q_1(s_1)$. Therefore, the asset price is $p_0 = q_1(s_1)$ because agent 1 is buying the asset in a positive amount by Lemma A5. ■

Proof of Corollary 2. By Theorem 1, there are as many as n different haircuts for the same collateral asset in any network equilibrium. For the second statement, recall the contract price equation (12)

$$q_j(d) = q_{j+1}(s_{j+1}) + \frac{E_j\left[\min\left\{1,\frac{d}{p_1}\right\}-\min\left\{1,\frac{s_{j+1}}{p_1}\right\}-\frac{\partial\Psi_{j+1,j}(C)}{\partial c_{j+1,j}}\left[1-\frac{s_{j+1}}{p_1}\right]^+\mathbb{1}\{j+1\in B(\epsilon)\}\right]}{E_j\left[\frac{1}{p_1}\right]}$$

from Theorem 1. Then, the gross interest rate for the contract borrowing from j becomes

$$\frac{s_j}{q_j(s_j)} = \frac{s_j}{q_{j+1}(s_{j+1}) + \frac{E_j\left[\min\left\{1,\frac{s_j}{p_1}\right\}-\min\left\{1,\frac{s_{j+1}}{p_1}\right\}-\frac{\partial\Psi_{j+1,j}(C)}{\partial c_{j+1,j}}\left[1-\frac{s_{j+1}}{p_1}\right]^+\mathbb{1}\{j+1\in B(\epsilon)\}\right]}{E_j\left[\frac{1}{p_1}\right]}}$$

The second statement is true if

$$\frac{s_j}{q_j(s_j)} < \frac{s_{j+1}}{q_{j+1}(s_{j+1})} \tag{A11}$$

holds. Then, applying equation (12) on both sides of (A11) yields

$$\frac{s_j}{q_{j+1}(s_{j+1}) + \frac{E_j\left[1-\min\left\{1,\frac{s_{j+1}}{p_1}\right\}-\frac{\partial\Psi_{j+1,j}(C)}{\partial c_{j+1,j}}\left[1-\frac{s_{j+1}}{p_1}\right]^+\mathbb{1}\{j+1\in B(\epsilon)\}\right]}{E_j\left[\frac{1}{p_1}\right]}} < \frac{s_{j+1}}{q_{j+1}(s_{j+1})}$$

and the following algebra yields

$$s_jq_{j+1}(s_{j+1}) < s_{j+1}q_{j+1}(s_{j+1}) + \frac{s_{j+1}}{E_j\left[\frac{1}{p_1}\right]}E_j\left[1-\min\left\{1,\frac{s_{j+1}}{p_1}\right\}-\frac{\partial\Psi_{j+1,j}(C)}{\partial c_{j+1,j}}\left[1-\frac{s_{j+1}}{p_1}\right]^+\mathbb{1}\{j+1\in B(\epsilon)\}\right]$$

$$(s_j - s_{j+1})q_{j+1}(s_{j+1}) < \frac{s_{j+1}}{E_j\left[\frac{1}{p_1}\right]}E_j\left[1-\min\left\{1,\frac{s_{j+1}}{p_1}\right\}-\frac{\partial\Psi_{j+1,j}(C)}{\partial c_{j+1,j}}\left[1-\frac{s_{j+1}}{p_1}\right]^+\mathbb{1}\{j+1\in B(\epsilon)\}\right]$$

$$\frac{E_j\left[\frac{s_j - s_{j+1}}{p_1}\right]}{E_j\left[1-\min\left\{1,\frac{s_{j+1}}{p_1}\right\}-\frac{\partial\Psi_{j+1,j}(C)}{\partial c_{j+1,j}}\left[1-\frac{s_{j+1}}{p_1}\right]^+\mathbb{1}\{j+1\in B(\epsilon)\}\right]} < \frac{s_{j+1}}{q_{j+1}(s_{j+1})}.$$

Applying equation (12) again to the last inequality becomes

$$\frac{E_j\left[\dfrac{s_j - s_{j+1}}{p_1}\right]}{E_j\left[1-\min\left\{1,\dfrac{s_{j+1}}{p_1}\right\}-\dfrac{\partial\Psi_{j+1,j}(C)}{\partial c_{j+1,j}}\left[1-\dfrac{s_{j+1}}{p_1}\right]^+ \mathbb{1}\{j+1\in B(\epsilon)\}\right]}$$
$$<\frac{E_{j+1}\left[\dfrac{s_{j+1}}{p_1}\right]}{q_{j+2}(s_{j+2})E_{j+1}\left[\dfrac{1}{p_1}\right]+E_{j+1}\left[1-\min\left\{1,\dfrac{s_{j+2}}{p_1}\right\}-\dfrac{\partial\Psi_{j+2,j+1}(C)}{\partial c_{j+2,j+1}}\left[1-\dfrac{s_{j+2}}{p_1}\right]^+ \mathbb{1}\{j+2\in B(\epsilon)\}\right]} \tag{A12}$$

and the right-hand side of (A12) is larger than

$$\frac{E_{j+1}\left[\dfrac{s_{j+1}}{p_1}\right]}{E_{j+1}\left[\dfrac{s_{j+2}}{p_1}\right]+E_{j+1}\left[1-\min\left\{1,\dfrac{s_{j+2}}{p_1}\right\}-\dfrac{\partial\Psi_{j+2,j+1}(C)}{\partial c_{j+2,j+1}}\left[1-\dfrac{s_{j+2}}{p_1}\right]^+ \mathbb{1}\{j+2\in B(\epsilon)\}\right]}. \tag{A13}$$

If s_j and s_{j+1} are close enough to each other, s_{j+2} is small enough, the probabilities of bankruptcy for $j+1$ and $j+2$ are similar to each other, and the price is almost always the fair price (for example, because n is relatively large), then the left-hand side of (A12) is smaller than (A13). Therefore, the inequality (A11) can hold, and the statement is true. ■

Proof of Theorem 2. The first three properties come directly from Theorem 1. The fourth property comes from the indifference equation for borrower j, who has to be indifferent between borrowing cash from i and k if j is borrowing from the two in a positive amount. The fifth property is simply from the budget constraint and contract prices.

Now I show that an equilibrium satisfying those properties exists in the following steps.

Step 1. (Space of Collateralized Debt Networks) Fix D as a lower triangular matrix with $d_{ij} = s_i$ for any $i > j$. Consider a class of networks $\mathcal{C}$ such that every $C \in \mathcal{C}$ is a lower triangular matrix with column sums $C_i \geq C_j$ for any $i < j$ so that C satisfies the intermediation order for the fixed D. The set $\mathcal{C}$ is a convex and compact subset of the Euclidean space.

Step 2. (Iterative Optimization Mapping) Let $V : \mathcal{C} \times \mathbb{R}^n_+ \to \mathcal{C} \times \mathbb{R}^n_+$ be a mapping from a network to networks—that is, agents compute $p_0, \tilde{p}_1, q$ and counterparty risk distribution ω given the first network C^0 and asset holdings a^1 in $t = 1$, and V generates the agents' optimal network formation decisions C_{C^0,a^1} and asset holdings $a^1_{C^0,a^1}$ as best responses with the new market clearing price p_0^*. The *iterative optimization* problem for each agent under V given C^0 and a^1 is

$$\max_{e_j^1,\{c_{ij}\}_{i\in N}} E_{j|C^0,a^1}\left[\left(e_j^1-\epsilon_j+a_j^1p_1+\sum_{k<j}c_{jk}\min\{s_j,p_1\}\right.\right.$$
$$\left.\left.-\sum_{i>j}c_{ij}\min\{s_i,p_1\}-\sum_{i\in B(\epsilon)}\Psi_{ij}(C)[p_1-s_i]^+\right)\frac{s_j}{p_1}\right]^+$$

$$\text{s.t.} \tag{A14}$$
$$a_j^1+\sum_{k<j}c_{jk}\geq\sum_{i>j}c_{ij},$$
$$e^0=e_j^1-\sum_{i>j}c_{ij}q_{i|C^0,a^1}(s_i)+\sum_{k<j}c_{jk}q_{j|C^0,a^1}(s_j)+a_j^1p_{0|C^0,a^1},$$
$$A-\sum_{k<j}a_k^1\geq a_j^1,$$

where the amount of lending c_{jk}, and the amount of asset purchase a_j^1 are given by the optimization decisions of the previous agents $k<j$ and the only macro variable p_0^* is determined endogenously. V solves the agents' optimization problem iteratively starting from agent 1. Fixing the previous agents' decisions, which is by Lemma 1, automatically satisfies the market clearing condition for each contract.

Next, I show that this V is a function because the optimal portfolio decision for (A14) is unique for each agent holding other agents' decisions fixed. There are two different dimensions of choice—how much cash to hold and how to borrow from different counterparties.

First, each agent decides on how much cash to hold. For any given C^0 and a^1, Lemma 1 applies so every agent is holding a positive amount of cash. Decrease in e_j^1 leads to higher expected cash return because there will be less amount of cash for j under ϵ with liquidity constrained price. Therefore, for an optimal portfolio of counterparty borrowing, the cash return should equate the return from intermediation.

Second, each agent decides on how to borrow from different counterparties. For a given lending decided by previous agents, an agent's optimal decision is unique due to linearity of payoffs—that is, $(q_{i|C^0,a^1}-\min\{s_i,p_1\})$—and convexly increasing lender default loss of (5). In particular, the following equation is derived from the first order conditions

$$E_j\left[\left(q_{i|C^0,a^1}-\min\{s_i,p_1\}-\mathbb{1}\{i\in B(\epsilon)\}\frac{\partial\Psi_{ij}(C)}{\partial c_{ij}^*}[p_1-s_i]^+\right)\frac{s_j}{p_1}\right]-\mu=0,$$

where μ is the Lagrangian multiplier for the collateral constraint.[20] Hence, for the fixed asset and contract purchase decision, agent 1's optimal borrowing portfolio should at least equate the cash return and intermediation return. Fixing up to agent $i-1$'s decision, agent i's collateral constraint

[20] Note that there is no tradeoff between cash holdings and intermediation for a fixed purchase decisions.

is determined and the problem is isomorphic to agent 1's problem and the solution is unique as well. Then, the iterative optimization mapping V is a function.

Step 3. (Asset Holdings Determination) The last object is to determine the new asset holdings vector. First, consider whether the given asset holdings vector a^1 clears the market while satisfying the optimality of each agent. Suppose that agent 1 clears the market and agent 1's cash return $E_1[s/p_1]$ does not exceed the return from leveraged purchase of the asset—that is,

$$\frac{E_1\left[\left(p_1 - \min\{s_i, p_1\} - \mathbb{1}\left\{i \in B(\epsilon)\right\} \frac{\partial \Psi_{i1}(C)}{\partial c^*_{i1}}[p_1 - s_i]^+\right)\frac{s_1}{p_1}\right]}{p_{0|C^0,a^1} - q_{i|C^0,a^1}} \geq E_1\left[\frac{s_1}{p_1}\right], \tag{A15}$$

for a $i \in N$ with $c^*_{i1} > 0$. Then, the given price $p_{0|C^0,a^1}$ and asset holdings a^1 solves the asset market clearing condition and optimality, and the asset price remains to be $p_{0|C^0,a^1}$.

Now suppose that inequality (A15) does not hold—that is, even the best portfolio choice of agent 1 cannot make intermediation return equate agent 1's cash return. Then, the asset price p^*_0 should be updated to make the inequality (A15) hold. If $p^*_0 < q_{2|C^0,a^1}$, then the price should be $p^*_0 = q_{2|C^0,a^1}$ following Theorem 1, and the asset holdings should be adjusted to $a^{1*}_1 < A$ to make (A15) hold as equality—by increasing cash holdings and by decreasing total collateral exposure.

With this residual supply of assets, $A - a^{1*}_1$, the asset price p_0 should change to $p^*_0 = q_{2|C^0,a^1}$ so that agent 2 (who is the next natural buyer by Lemma A5) will purchase the assets as well. Agent 2's problem becomes isomorphic to agent 1's original problem with the total supply of collateral $A - a^{1*}_1 + c^*_{21}$. Then, iterate the same procedure and check whether the optimal a^{1*}_2 clears $A - a^{1*}_1$ or not.

To complete the structure of induction, suppose that agent $k-1$ solved for the asset purchase problem. For agent k, the residual asset supply is given by $A - \sum_{i=1}^{k-1} a^{1*}_i$ for a given optimal portfolio decisions $C^*_1, C^*_2, \ldots, C^*_{k-1}$. If the residual asset supply is positive, then the new prices are $p_0 = q_2 = \cdots = q_{k|C^0,a^1}$. Agent k solves the optimal portfolio problem with the given supply of collateral $A - \sum_{i=1}^{k-1}\left(a^{1*}_i - c_{ki}\right)$. If the return inequality

$$\frac{E_k\left[\left(p_1 - \min\{s_i, p_1\} - \mathbb{1}\left\{i \in B(\epsilon)\right\} \frac{\partial \Psi_{ik}(C)}{\partial c^*_{ik}}[p_1 - s_i]^+\right)\frac{s_k}{p_1}\right]}{p^*_0 - q_{i|C^0,a^1}} \geq E_k\left[\frac{s_k}{p_1}\right], \tag{A16}$$

is satisfied, then the market is cleared. Otherwise, adjust the asset price (and k's contract price for the promise s_k) to p^*_0 that satisfies (A16) with equality. If $p^*_0 \geq q_{k+1|C^0,a^1}$, the step is done. Otherwise, update price to $p^*_0 = q_{k+1|C^0,a^1}$, derive a^{1*}_k, and then iterate problem for agent $k+1$ again. There will eventually be a unique solution $p^*_0 \geq 0$ that clears the market because the asset price is decreasing over the procedure while the left-hand side of (A16) is decreasing in the asset price.

Step 4. (Continuity of Macro Variables) For given contract and asset prices, a change in borrowing and lending affect both the price fluctuations $\tilde{p}_1$ and counterparty risks ω. I show that both are changing continuously in C^0. First, consider a fixed set of liquidity shocks with the same bankruptcy set $B(\epsilon) = B(\epsilon')$ for $\epsilon, \epsilon' \in \mathcal{E}$. A change in C increases or decreases price continuously in (7). Now suppose that for a fixed ϵ, $B(\epsilon|C) \neq B(\epsilon|C')$ for two different collateral matrices with $\|C - C'\| < \delta$. There will be an additional jump in bankruptcy cost $\beta_l(C)$ for $l \in B(\epsilon|C)\backslash B(\epsilon|C')$. However, the measure of such liquidity shock realizations is bounded by

$$\mathcal{G}(\delta) \equiv \max_{x \in R^+} \left[G_\Sigma \left(x + \delta \max_{i,j \in D} d_{ij} \right) - G_\Sigma(x) \right],$$

where G_Σ is the distribution function of $g_\Sigma = g_1 * g_2 * \cdots * g_n$ that is the convolution of density functions g_j of liquidity shock for each agent j. Therefore, for any small ι, there always exists a δ that can make $\mathcal{G}(\delta)\beta_l(C) < \iota$ because G is differentiable over $[0, \bar{\epsilon}]$. Then, in any agent's perspective, the expected price is changing continuously over C^0. Similarly, $\omega_{ij}(C)$ and $\Psi_{ij}(C)$ are changing continuously in C. Therefore, price fluctuations as well as counterparty risks are continuously changing in C^0.

Step 5. (Continuity of V) Because the distribution of prices and counterparty risks are changing continuously, the contract prices and expected utility are also changing continuously. Since the choice set under the constraints are compact and continuous in C and the maximization problem is a function, the optimal portfolio choices are also continuous in C by Berge's maximum theorem. Then, V is also continuous in C.

Step 6. (Fixed Point Theorem) Since V is a continuous mapping that maps a convex compact subset of the Euclidean space to itself, there exists a C^* such that $V(C^*) = C^*$ by the Brouwer fixed point theorem.

Now the rest of the proof is simply applying the results with $q(d)$ and p_0 from Theorem 1 into market clearing conditions. Also, the nominal wealth is determined by the combination of budget constraints and market clearing conditions. ■

Proof of Proposition 3. First, note that any agent j faces higher counterparty risk from agent i than that from $k > i$ in a network equilibrium, because otherwise j will prefer to borrow more from agent i rather than borrowing from agent k by Theorem 2 and Lemma A2.

Suppose that agent j diversifies its counterparties across i and k such that $j < i < k$ with k being $E_k[\min\{s_k, p_1\} - q_k(s_k)] > 0$, which exists by Lemma A5 (e.g. agent n). Note that $E_j[\min\{s_k, p_1\} - q_k(s_k)] \geq E_k[\min\{s_k, p_1\} - q_k(s_k)]$ for any $j < k$. Thus, a marginal change of portfolio by shifting the borrowing from i to k will decrease the total counterparty risk of j and the new allocation is a diversification of agent j from C. Denote this new collateral matrix as $\tilde{C}$ such that a marginal change from C_j toward $\tilde{C}_j$ is a diversification of j. Let $C_j(t)$ denote a vector-valued function such that

$$C_j(t) = \begin{pmatrix} c_{j1} + t(\tilde{c}_{j1} - c_{j1}) \\ c_{j2} + t(\tilde{c}_{j2} - c_{j2}) \\ \vdots \\ c_{jn} + t(\tilde{c}_{jn} - c_{jn}) \end{pmatrix},$$

therefore, $C_j'(t)$ is the directional derivative of C_j toward $\tilde{C}_j$. Also note that $C_j'(t)$ is possible because there are slacks in budget constraints of all agents by Lemma 1.

This marginal change will have four effects on the expected asset price, which is inversely related to the systemic risk of the allocation. First, the diversification changes the bankruptcy probability of agent j, who becomes safer after the diversification because of the decrease in expected counterparty losses. Recall that $\sum_{i \in N} \Psi_{ij}(C)[p - s_i]^+ \mathbb{1}\{i \in B(\epsilon)\}$ is the counterparty cost side of j that determines the likelihood of bankruptcy. Now for the marginal change $C_j'(t)$ in the network, there will be a change in counterparty default risk $\nabla \omega_{jk}(C_j) \cdot C_j'(t)$ for any $k < j$, which is positive by the definition of diversification of j. This effect will always increase the expected asset price under any agent's expectation. Second, the diversification increases the aggregate cash holdings through decrease in leverage. Again, this effect will increase the expected asset price under any agent's expectation. Third, the change will increase the expected nominal wealth of agent k, who is lending more to j after the diversification. This is because of the changes in the payment received in $t = 1$ minus the payment made in $t = 0$ for agent k is $\partial c_{kj}(s_k - q_k(s_k)) > 0$, when $p_1 \geq s_k$. Therefore, lender k is less likely to go bankrupt whenever $p_1 \geq s_k$.

Fourth, there will be a change in lender default loss functions $\Psi_{i\cdot}$ and $\Psi_{k\cdot}$. The effect on Ψ_{il} for any l will be negative as agent j is reducing c_{ij}, so the expected price will increase. However, the effect on Ψ_{kl} for any l with $c_{kl} > 0$ will be positive as the pool of collateral exposures to lender k increases. For the optimists $l < j$, this effect is always smaller than the direct decrease in counterparty risks of j as they would be borrowing more from j rather than k. For the pessimists $l > j$, it requires more detailed comparison to confirm the effect on their expected asset prices. In particular, I will compare the third effect to the fourth effect for the pessimist agents $l > j$, and show that the increase in the expected asset price from the third effect dominates the decrease in the expected asset price from the fourth effect.

The counterparty risk from k that $l > j$ is facing is $E_l\left[\Psi_{kl}(C)[p_1 - s_k]^+ \mathbb{1}\{k \in B(\epsilon)\}\right]$, which will increase by $E_l\left[\frac{\partial \Psi_{kl}(C)}{\partial c_{kj}}[p_1 - s_k]^+ \mathbb{1}\{k \in B(\epsilon)\}\right]$. By Assumption 1, this is smaller than $E_l\left[\frac{\partial \Psi_{kl}(C)}{\partial c_{kl}}[p_1 - s_k]^+ \mathbb{1}\{k \in B(\epsilon)\}\right]$. From (12),

$$E_l\left[\frac{s_l}{p_1}\left([p_1 - s_k]^+ - \frac{\partial \Psi_{k,l}(C)}{\partial c_{kl}}[p_1 - s_k]^+ \mathbb{1}\{k \in B(\epsilon)\}\right)\right] = E_l\left[\frac{s_l}{p_1}\left(q_l(s_l) - q_k(s_k)\right)\right],$$

and $E_l\left[\frac{s_l}{p_1}[p_1 - s_k]^+\right] > E_l\left[\frac{s_l}{p_1}\left(q_l(s_l) - q_k(s_k)\right)\right] > E_l\left[\frac{s_l}{p_1}\left(q_l(s_l) - \min\{s_k, p_1\}\right)\right]$ holds by

Lemma A5 and the assumption on k. Then,

$$E_l\left[\frac{s_l}{p_1}\left(-\frac{\partial\Psi_{k,l}(C)}{\partial c_{kl}}[p_1-s_k]^+\mathbb{1}\{k\in B(\epsilon)\}\right)\right]=E_l\left[\frac{s_l}{p_1}\left(q_l(s_l)-q_k(s_k)-[p_1-s_k]^+\right)\right]$$
$$<E_l\left[\frac{s_l}{p_1}\left(\min\{s_k,p_1\}-q_k(s_k)\right)\right]$$

Therefore, the third effect from diversification $\partial c_{kj}(\min\{s_k,p_1\}-q_k(s_k))$ dominates the fourth effect $\frac{\partial\Psi_{kl}(C)}{\partial c_{kj}}[p_1-s_k]^+\mathbb{1}\{k\in B(\epsilon)\}$ under agent l's expectation. ■

Proof of Theorem 3. The structure of the proof is as follows. First, I show that the direct increase in counterparty risk increases the weight of counterparty risk and decreases the benefit of leverage. Therefore, in the tradeoff between counterparty risk and leverage, agents borrow more from more pessimistic lenders to diversify their counterparties more. This shift will lower the overall leverage and more so for the optimistic agents as their willingness to pay decreases furthermore on top of the direct increase of counterparty risk. Lower leverage will make the asset price lower and increase the overall cash holdings. Finally, agents have even more incentives to diversify their lenders as prices fluctuate less and the lender default is even more likely and more severe.

Suppose that idiosyncratic counterparty risk increases for everyone (for example, θ_i increases to $\tilde{\theta}_i>\theta_i$ for every $i\in N$). There are two directions of response to this increased counterparty risk and price fluctuations—increase in cash holdings and increase in diversification. By equation (12), the function for contract price becomes

$$q_j(d)=q_{j+1}(s_{j+1})+\frac{E_j\left[\min\left\{1,\frac{d}{p_1}\right\}-\min\left\{1,\frac{s_{j+1}}{p_1}\right\}-\frac{\partial\Psi_{j+1,j}(C)}{\partial c_{j+1,j}}\left[1-\frac{s_{j+1}}{p_1}\right]^+\mathbb{1}\{j+1\in B(\epsilon)\}\right]}{E_j\left[\frac{1}{p_1}\right]}$$

By Theorem 1, only $d=s_i$ will be traded for a lending from $i\in N$ in any equilibrium. Any change in the terms related to $q(s_j)$ has a direct effect on $q(s_i)$ in linear terms for any $i<j$ by the recursive equation

$$q(s_i)=q(s_j)+\sum_{k=i+1}^{j-1}\frac{E_k\left[1-\min\left\{1,\frac{s_{k+1}}{p_1}\right\}-\frac{\partial\Psi_{k+1,k}(C)}{\partial c_{k+1,k}}\left[1-\frac{s_{k+1}}{p_1}\right]^+\mathbb{1}\left\{k+1\in B(\epsilon)\right\}\right]}{E_k\left[\frac{1}{p_1}\right]}.$$

As the counterparty risk increases, each agent's subjective cash return increases. But, the increase in cash return would be larger for more optimistic agents as

$$\Delta E_1\left[\frac{s}{p_1}\right]>\Delta E_2\left[\frac{s}{p_1}\right]>\cdots>\Delta E_n\left[\frac{s}{p_1}\right],$$

because more optimistic agents value the asset more given the same liquidity constrained price, where Δ denotes the change of a variable. For any agent $k < j$, prices relevant to cashflow of the leveraged contracts are bounded below by the subject belief of the lender $k+1$, which is s_{k+1}. However, the return from cash holdings, $s_k E_k\left[1/p_1\right]$ is not bounded by any price. The ratio between the changes of the two terms is increasing in k as the lower bound of the price distribution becomes smaller—that is,

$$\frac{\Delta E_k\left[1-\min\left\{1,\frac{s_{k+1}}{p_1}\right\}-\frac{\partial\Psi_{k+1,k}(C)}{\partial c_{k+1,k}}\left[1-\frac{s_{k+1}}{p_1}\right]^+\mathbb{1}\left\{k+1\in B(\epsilon)\right\}\right]}{\Delta E_k\left[\frac{1}{p_1}\right]}$$
$$<\frac{\Delta E_{k+1}\left[1-\min\left\{1,\frac{s_{k+2}}{p_1}\right\}-\frac{\partial\Psi_{k+2,k+1}(C)}{\partial c_{k+2,k+1}}\left[1-\frac{s_{k+2}}{p_1}\right]^+\mathbb{1}\left\{k+2\in B(\epsilon)\right\}\right]}{\Delta E_{k+1}\left[\frac{1}{p_1}\right]}.$$

Then, there will be more tightening of interest rates for more optimistic agents as

$$-\Delta q_1(s_1) > -\Delta q_2(s_2) > \cdots > -\Delta q_{n-1}(s_{n-1}) > -\Delta q_n(s_n).$$

Thus, changes in expected payments from a more optimistic lender are lower than the changes in the amount of lending (price of the contract) from a more optimistic lender. Therefore, agent $i < n-1$ will have a greater decrease in expected return of borrowing from $i+1$ compared to that of borrowing from $i+2$ as

$$-\Delta R_i^{i+1} > -\Delta R_i^{i+2},$$

for the same C_i, where R_i^j denotes the return of agent i from borrowing from j as

$$R_i^j \equiv \frac{s_i}{q(s_i)-q(s_j)}E_i\left[\min\left\{1,\frac{s_i}{p_1}\right\}-\min\left\{1,\frac{s_j}{p_1}\right\}-\frac{\partial\Psi_{ji}(C)}{\partial c_{ji}}\left[1-\frac{s_j}{p_1}\right]^+\mathbb{1}\left\{j\in B(\epsilon)\right\}\right].$$

Hence, the higher leverage of borrowing from a more optimistic lender cannot justify the higher counterparty risk. Agent i will decrease $c_{i+1,i}$, and instead borrow from more pessimistic agents, which implies more links. Also, the reuse of collateral (weakly) decreases because of the decrease in $c_{i+1,i}$ as well as tightening collateral constraints for the subsequent agents $i+1, i+2, \ldots, n$.[21]

[21]reuse of collateral in a collateral matrix C can be measured by the collateral multiplier defined below—the volume of total collateral posted divided by the stock of source collateral as

$$\mathcal{CM}(C) \equiv \frac{\sum_{i\in N}\sum_{j\neq i} c_{ij}}{\sum_{j\in N} a_j^1}.$$

This collateral multiplier represents the volume of reuse of collateral within the network. For example, if the network C is a single-chain network using all of the source collateral repeatedly, then the collateral multiplier of C is $n-1$ because $c_{21} = c_{32} = \cdots = c_{n,n-1} = A$ and $\mathcal{CM}(C) = (c_{21} + c_{32} + \cdots + c_{n,n-1})/A = n-1$. If the network C is a completely diversified multi-chain network, then $c_{21} = c_{31} = \cdots c_{n1} = n/n{-}1$, $c_{32} = \cdots = c_{n2} = (n/n{-}1)/(n{-}2), \ldots, c_{43} = \cdots =$

Also, this shift in borrowing pattern on top of the lower contract prices will lower the overall leverage. More optimistic agents will lend even less as their willingness to pay decreases furthermore on top of the direct increase of counterparty risk. Lower leverage will make the asset price in $t = 0$, p_0 lower and increase the overall cash holdings in the economy, $\sum_{i \in N} e_i^1$. This change will increase the asset price in $t = 1$. The increase in expected asset price due to the change in borrowing pattern will make the price more likely to be greater than the promise. Then, agents are more likely to pay lender default loss. Agents have even more incentives to diversify their lenders.

So, for the same collateral exposure, ω increases because of heightened idiosyncratic risk, whereas the leverage rather decreases. This will make agents diversify their borrowing more. The change in borrowing pattern will make the macro variable (asset price) more stable and rather increase the counterparty risk concern because borrowers are more willing to retrieve their collateral. Therefore, the shift in the distribution of the asset price makes agents diversify even more in the new equilibrium. ■

$c_{n3} = (n/n-1+(n/n-1)/(n-2))/(n-3), \ldots c_{n,n-1} = n/(n-1)+n/((n-1)(n-2))+n/((n-1)(n-3))+n/((n-1)(n-2)(n-3)) + \cdots + n/((n-1)(n-2)\cdots(n-n+2))$, with $\mathcal{CM}(C) = 2\left(\frac{1+2+2\cdot 3+2\cdot 3\cdot 4\cdots+2\cdot 3\cdots(n-1)}{(n-1)(n-2)\cdots 2\cdot 1}\right) < n-1$. This measure is consistent with the velocity of collateral in Singh (2017) and the collateral multiplier in Infante et al. (2018). The collateral multiplier is also an approximate measure of the average length of the lending chain in the network (Singh, 2017; Infante et al., 2018).

B. Omitted Results

This section contains omitted results and proofs mentioned in the main text and the appendix of the paper.

B.1. Comparative Statics of Payment Equilibrium

The comparative statics here focus on the change in the network structure while holding the agents' cash holdings the same. Therefore, I define the concept of cash compensation to fix the effective cash holdings after the change in the debt matrix. Define $\hat{e}^1$ as the *equivalent cash compensation of* $(\hat{C}, \hat{D})$ *from* e^1, if $\hat{e}^1$ compensates the cash holdings for the difference in total payments as

$$\hat{e}_j^1 = e_j^1 - \sum_{i \in N}(c_{ij} - \hat{c}_{ij})d_{ij} + \sum_{k \in N}(c_{jk} - \hat{c}_{jk})d_{jk} - \sum_{i \in N}(d_{ij} - \hat{d}_{ij})c_{ij} + \sum_{k \in N}(d_{jk} - \hat{d}_{jk})c_{jk}$$

for all $j \in N$.

Proposition B1 (Payment Equilibrium Comparative Statics)**.** *Let* (m^*, p^*) *be the payment equilibrium for a given period-1 economy with collateralized debt network* (C, D)*.*

1. *Suppose the network changes to* $(\hat{C}, D)$ *that is under intermediation under and* $\hat{c}_{ij}$ *that is less (greater) than or equal to* c_{ij} *for any* $i, j \in N$ *with strict inequality for at least one pair. Also, suppose that the cash holdings are* $\hat{e}^1 > 0$*, which is an equivalent cash compensation of* $(\hat{C}, D)$ *from* $e^1 > 0$*. Then, the expected asset price* $E[\tilde{p}^*]$ *is greater (less) than or equal to* $E[p^*]$ *for any distribution of* s*.*

2. *Suppose the asset payoff* $\tilde{s}$ *is greater (less) than* s*. Then, the equilibrium price* $\tilde{p}^*$ *under* $\tilde{s}$ *is greater (less) than* p^* *under* s*, and the number of bankrupt agents under* $\tilde{s}$ *is less than that under* s*.*

3. *Suppose the common liquidity shock distribution* G *becomes* $\tilde{G}$ *that (is) first order stochastically dominates (dominated by)* G*. Then, the expected equilibrium price* $E[\tilde{p}^*]$ *is less (greater) than* $E[p^*]$ *for any distribution of* s*.*

4. *Suppose the cash holdings change to* $\tilde{e}^1$ *that is* $\tilde{e}_j^1$ *is greater (less) than* $e_j^1 > 0$ *for every* $j \in N$*. Then, the expected equilibrium price* $E[\tilde{p}^*]$ *is greater (less) than or equal to* $E[p^*]$ *for any distribution of* s*.*

Proof of Proposition B1.

1. Consider the case that collateral exposure decreased. First, I show that the cash compensation does not decrease the expected asset price.

Case 1. First, consider the agents who only lend and do not borrow from another agent or purchase the asset. From (2), compensation of cash holdings will always increase the wealth of the pure lenders as

$$\begin{aligned}\hat{m}_j(s,\epsilon) &= \hat{e}_j^1 - \epsilon_j + \sum_{i\in N} \hat{c}_{ji}\min\{p, d_{ji}\} \\ &= e_j^1 - \epsilon_j + \sum_{i\in N} \hat{c}_{ji}\min\{p, d_{ji}\} + \sum_{i\in N}(c_{ji} - \hat{c}_{ji})d_{ji} \\ &> e_j^1 - \epsilon_j + \sum_{i\in N} c_{ji}\min\{p, d_{ji}\} = m_j(s,\epsilon),\end{aligned}$$

for the same realization (s, ϵ).

Case 2. For the second case, consider an intermediating agent $j \in N$ who reuses the collateral and have the collateral constraint binding. By the intermediation order, a decrease in lending should always correspond to a decrease in borrowing. Therefore, the compensation does not decrease the wealth of a purely intermediating agent as

$$\begin{aligned}\hat{m}_j(s,\epsilon) =& \hat{e}_j^1 - \epsilon_j + a_j^1 p + \sum_{i\in N}\left(\hat{c}_{ji}\min\{p, d_{ji}\} - \hat{c}_{ij}\min\{p, d_{ij}\}\right) - \sum_{i:m_i<0} \Psi_{ij}(\hat{C})[p - d_{ij}]^+ \\ =& e_j^1 - \epsilon_j + a_j^1 p + \sum_{i\in N}\left(\hat{c}_{ji}\min\{p, d_{ji}\} - \hat{c}_{ij}\min\{p, d_{ij}\}\right) - \sum_{i:m_i<0} \Psi_{ij}(\hat{C})[p - d_{ij}]^+ \\ & - \sum_{i\in N}(c_{ij} - \hat{c}_{ij})d_{ij} + \sum_{i\in N}(c_{ji} - \hat{c}_{ji})d_{ji} \\ \geq& e_j^1 - \epsilon_j + a_j^1 p + \sum_{i\in N}\left(c_{ji}\min\{p, d_{ji}\} - c_{ij}\min\{p, d_{ij}\}\right) - \sum_{i:m_i<0} \Psi_{ij}(C)[p - d_{ij}]^+,\end{aligned}$$

where the last inequality holds by the intermediation order.

Case 3. For the last case, consider an agent $j \in N$ who is either purchasing the asset ($a_j^1 > 0$) or intermediating but the collateral constraint of j is not binding. Agent j could possibly have lower cash holdings after the cash compensation in a state that the market price for the uncertainty realization (s, ϵ) resulted in $p_1 < d_{ij}$ for some $i \in N$. However, such borrowers are defaulting in such states anyway, so the cash transfer either does not affect the total cash holdings or, rather, increases the total cash holdings by preventing j's lenders from going bankrupt. Finally, this lowering of $m_j(p|s, \epsilon)$'s wealth could make agent j more likely to go bankrupt and inflict lender default loss to $\mathcal{V}_j$. However, by the intermediation order, agents who borrows from agent j shall default on their debt whenever agent j defaults. Therefore, the increased probability of j's bankruptcy does not lead to an increase in expected lender default.

Finally, I show that the new collateral matrix will increase the expected asset price by lowering the counterparty contagion. Since the coefficients on prices are lower, agent j's wealth is less susceptible to price change. Furthermore, j faces lower lender default loss by assumption 1 and the same or less probability of second-order bankruptcy for the same state realizations by Proposition 2. Then, both the price and counterparty channels of contagion decrease, and there will be less states with underpricing so that $E[\tilde{p}^*] \geq E[p^*]$ for any distribution of s.

Now consider the opposite case, increase in collateral exposure. The reverse cash compensation decreases the ex post wealth of the pure lenders. The cash compensation does not affect other agents as in the first part of the proof. Finally, the new collateral matrix increases the counterparty contagion as the coefficients for lender default $\Psi_{ij}(C)$ weakly increase for any $i, j \in N$. Therefore, the expected price decreases for any distribution of s.

2. If the equilibrium price was $p < s$ in the original period-1 economy, then the increase in s does not have any effect. Now consider the case that $p = s$. From (8), an increase in s can increase p. Suppose that the bankruptcy set remains the same as $B(\epsilon|\tilde{s}) = B(\epsilon|s)$. Since the maximum payment equilibrium is unique by Proposition 1, there is no need to consider the case with the bankruptcy set larger than $B(\epsilon|s)$ if there is an equilibrium with $B(\epsilon|\tilde{s}) \subseteq B(\epsilon|s)$. If the equilibrium price remains the same as $p = s$, then the same market clearing condition holds only under (3) and this is the (trivial) new equilibrium with the same bankruptcy set. Finally, the only case left is the equilibrium with price $\tilde{p} > s$. If agents trade in $\tilde{p}$, $m_j(p)$ increases for each $j \in N$ by Lemma A1. Therefore, any agent who was not bankrupt under s does not go bankrupt under $\tilde{s}$ as well so $B(\epsilon|\tilde{s}) \subseteq B(\epsilon|s)$. By (3), the equilibrium price increases (up to $\tilde{s}$). The other direction follows the same argument.

3. The result follows immediately from Proposition 2.

4. For each realization of s and ϵ, $m_j(p|s, \epsilon)$ only increases (decreased) by $\tilde{e}_j^1 - e_j^1$ for any $j \in N$. Therefore, the equilibrium price increases (decreases) and the size of the bankruptcy set goes the opposite direction, amplifying the increase (decrease) by Proposition 2.

■

Now, I show how diversification of counterparties of an agent affects network contagion.

Proposition B2 (Diversification Externality)**.** *Let $(N, C, D, e^1, a^1, \cdot, \cdot, \Psi)$ be a period-1 economy, and $\dfrac{\partial \Psi_{ij}(C)}{\partial c_{ik}} = 0$ and $d_{ij} = d_{ik}$ for any $i, j, k \in N$. Suppose $\tilde{C}$ is a diversification of agent $j < n$ from C, and $\tilde{e}^1$ is the equivalent cash compensation of $(\tilde{C}, D)$ from e^1. Then, the expected payment equilibrium price $E_i[\tilde{p}^*]$ under $(N, \tilde{C}, D, \tilde{e}^1, a^1, \cdot, \cdot, \Psi)$ is greater than $E_i[p^*]$ of the original economy for any agent i who is not lending to j.*

Proof of Proposition B2.

If $j = n - 1$ or $c_{ij} > 0$ for only one $i \in N$, the statement holds immediately by statement 1 of Proposition B1 because the change is equivalent to decreasing the collateral matrix with equivalent cash compensation.

Now suppose $j < n-1$ and there are i, k with $i \neq k$ such that $c_{ij}, c_{kj} > 0$. If the change is simply decreasing both c_{ij} and c_{kj} simultaneously, then again the statement holds immediately by statement 1 of Proposition B1. Therefore, the only cases left to consider are the cases with c_{ij} and c_{kj} changing to different directions.

Suppose $\tilde{c}_{ij} < c_{ij}$ and $\tilde{c}_{kj} > c_{kj}$ without loss of generality. There will be three effects to consider: the direct counterparty effect, the cash holdings effect, and the intermediation effect.

First, ω_{jl} will decrease for any $l < j$ by the definition of diversification of agent j. This will in turn decrease the second-order bankruptcy of agent l and l's counterparties, so ω_{ml} decreases for m such that $l \in \mathcal{V}_m$.

Second, there will be no difference in counterparty risks and payments for agents other than the lenders to agent j in any payment equilibrium for a given (s, ϵ). This is because of cash compensation $\tilde{e}^1$ and the same face value of the debt for common lenders $d_{kj} = d_{kl}$ for any j, k, l. If borrower j does not default, the total cash payment plus cash holdings for agent k will be the same as in the original economy because $e^1_k - \tilde{e}^1_k = (\tilde{c}_{kj} - c_{kj})d_{kj}$. If the borrower j defaults, then the lender k may have lower wealth after the payment because $e^1_k - \tilde{e}^1_k = (\tilde{c}_{kj} - c_{kj})d_{kj} > (\tilde{c}_{kj} - c_{kj})p$. However, any agent l who is borrowing from k would have defaulted as well because $p > d_{kj} = d_{kl}$. Therefore, the increased likelihood of lender bankruptcy is irrelevant to other agents because there will be no relevant lender default losses for them.

Third, the possible change in intermediation pattern rather (weakly) increases equilibrium prices for any (s, ϵ) realized. If none of the collateral constraints are binding after the change to $\tilde{C}$, then there will be no additional effect to consider. Now suppose that the collateral constraint for agent i is binding because $\sum_{l \neq j} c_{il} + c_{ij} > \sum_{l \neq j} c_{il} + \tilde{c}_{ij}$. Then, agent i must borrow less from the set of lenders $\mathcal{V}_i$. This additional change is equivalent to decreasing the collateral matrix with equivalent cash compensation and only increases equilibrium price by statement 1 of Proposition B1 again. Hence, the change in the intermediation pattern will only increase the equilibrium price.

Finally, all these arguments for two agents $i, k \in \mathcal{V}_j$ can be applied to any other arbitrary set of agents lending to j. Therefore, the expected equilibrium price for agents other than agents lending to j will be larger than the original expected equilibrium price. ■

Figure B1 shows the numerical results that demonstrate comparative statics of the payment equilibrium in $t = 1$. Each panel of figure B1 shows the monotonic effect of the comparative static result.

Finally, I discuss the absence of comparative statics for many other possible directions that are common in the financial networks literature. The main reason is the complexity of the multi-dimensional collateralized debt networks. For example, one can consider an increase in interconnectedness by increasing the number of counterparties of an agent $j \in N$ while fixing the total amount of debt for agent j. The resulting price distribution depends on the exact contract terms d_{ij} for each $i \in N$, the holdings of cash and asset (e^1_i, a^1_i), and the liability structure $[c_{ki}, d_{ki}]_{k \in N}$ for each counterparty $i \in N$. If agent j was exclusively connected to an agent with very low probability of bankruptcy already, increasing the counterparties may rather increase the total expected

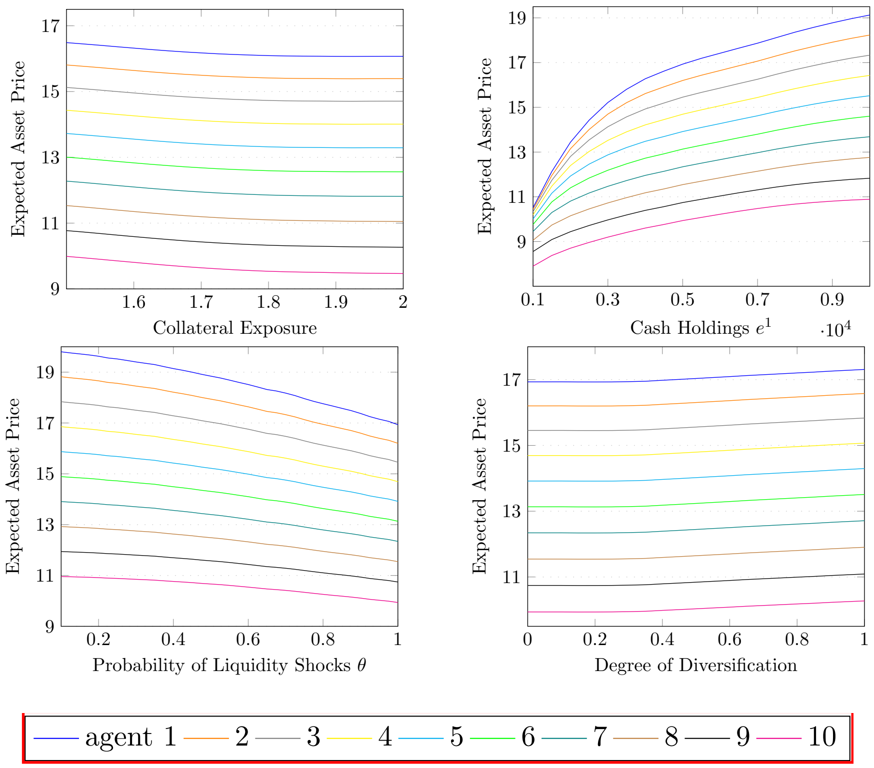

Figure B1: Numerical comparative statics results.

Note: Vertical axis for each graph represents the expected asset price of an agent and horizontal axis of each graph represents the level of parameter value for each comparative statics. Each line represents the subjective expected asset price of an agent. See the online appendix for the details of the numerical exercise.

counterparty risk of j. Therefore, there is no single sufficient statistic such as a single centrality measure that summarizes the systemic risk of a collateralized debt network.

B.2. Pareto Inefficiency of Network Equilibrium

In the main text, I showed that there are externalities of diversification in terms of lowering systemic risk under any agent's belief. The next result shows that the allocation in a network equilibrium can be improved by diversification and appropriate cash transfers. The only difference from the main setting is that I assume no cross-exposure effects on lender default losses for this result.

Proposition B3 (Lack of Diversification)**.** *Assume that* $\dfrac{\partial \Psi_{ij}(C)}{\partial c_{ik}} = 0$ *for any distinct* i, j, k. *Suppose that* $(C, D, e^1, a^1, p_0, \tilde{p}_1, q)$ *is a network equilibrium and there exists an agent* $j > 1$ *who is borrowing from more than two different lenders. Then, there exists an allocation that Pareto dominates the equilibrium allocation by diversifying the counterparties of agent* j *with cash transfers.*

Proof of Proposition B3. Suppose that agent j is borrowing from more than two distinct lenders. By 4 of Theorem 2 and Lemma A2 in the appendix,

$$
\begin{aligned}
&\frac{s_j}{q(s_i)} E_j \left[\min\left\{1, \frac{s_i}{p_1}\right\} + \frac{\partial \Psi_{ij}(C)}{\partial c_{ij}} \left[1 - \frac{s_i}{p_1}\right]^+ \mathbb{1}\{i \in B(\epsilon)\} \right] \\
=&\frac{s_j}{q(s_k)} E_j \left[\min\left\{1, \frac{s_k}{p_1}\right\} + \frac{\partial \Psi_{kj}(C)}{\partial c_{kj}} \left[1 - \frac{s_k}{p_1}\right]^+ \mathbb{1}\{k \in B(\epsilon)\} \right]
\end{aligned} \tag{B17}
$$

for any $i < k$ with $c_{ij}, c_{kj} > 0$. Agent j faces higher counterparty risk from agent i, because otherwise agent j will prefer to borrow more from agent i by Lemma A5. Thus, a marginal change of portfolio by shifting the borrowing from i to k will decrease the total counterparty risk of j. Then, there exists a direction from C_j such that a marginal change of C_j is a diversification of j. Consider such a marginal change from C_j toward $\tilde{C}_j$, which is a diversification of agent j from C_j. Let $C_j(t)$ denote a vector-valued function such that

$$
C_j(t) = \begin{pmatrix} c_{j1} + t(\tilde{c}_{j1} - c_{j1}) \\ c_{j2} + t(\tilde{c}_{j2} - c_{j2}) \\ \vdots \\ c_{jn} + t(\tilde{c}_{jn} - c_{jn}) \end{pmatrix},
$$

therefore, $C_j'(t)$ is the directional derivative of C_j toward $\tilde{C}_j$. Also note that $C_j'(t)$ is possible because there are slacks in budget constraints of all agents by Lemma 1.

From (B17), agent j's marginal cost of adjustment is

$$
E_j \left[\frac{s}{p_1} \begin{pmatrix} \dfrac{\min\{s_1, p_1\}}{q_1(s_1)} \\ \vdots \\ \dfrac{\min\{s_{j+1}, p_1\}}{q_{j+1}(s_{j+1})} \\ \vdots \\ \dfrac{\min\{s_n, p_1\}}{q_n(s_n)} \end{pmatrix} \right] \cdot C_j'(t) + \begin{pmatrix} \dfrac{\partial \Psi_{ij}(C)}{\partial c_{1j}(t)} \\ \vdots \\ \dfrac{\partial \Psi_{j+1,j}(C)}{\partial c_{j+1,j}(t)} \\ \vdots \\ \dfrac{\partial \Psi_{nj}(C)}{\partial c_{nj}(t)} \end{pmatrix} \circ \begin{pmatrix} \dfrac{\omega_{1j}(s_1)}{q_1(s_1)} \\ \vdots \\ \dfrac{\omega_{j+1,j}(s_{j+1})}{q_{j+1}(s_{j+1})} \\ \vdots \\ \dfrac{\omega_{nj}(s_n)}{q_n(s_n)} \end{pmatrix} \cdot C_j'(t) = 0,
$$

which is zero because of the optimality condition of agent j.

Recall that $\sum_{i \in N} \Psi_{ij}(C)[p - s_i]^+ \mathbb{1}\{i \in B(\epsilon)\}$ is the counterparty cost side of j that determines the likelihood of bankruptcy. Now for the marginal change $C_j'(t)$ in the network, there will be a change in counterparty default risk $\nabla \omega_{jk}(C_j) \cdot C_j'(t)$ for any $k < j$, which is positive by the definition of diversification of j.

By Lemma 1, the cash equivalent change in utility for agent $j - 1$ is

$$
\frac{\Psi_{j,j-1}(C) \nabla \omega_{j,j-1}(C_j) \cdot C_j'(t)}{E_{j-1} \left[\dfrac{s}{p_1} \right]}
$$

which is the cash equivalent compensation (willingness to pay) from $j-1$. For agent $j-2$,

$$\frac{\Psi_{j,j-2}(C)\nabla\omega_{j,j-2}(C)\cdot C_j'(t)+\Psi_{j-1,j-2}(C)\nabla\omega_{j-1,j-2}(C)\cdot C_j'(t)}{E_{j-2}\left[\dfrac{s}{p_1}\right]}$$

is the first- and second-order effect to $j-2$ that are all positive since $j-1$ only becomes safer as well. Similarly, the total cash equivalent compensation from agent 1 through $j-1$ for diversification of j will be

$$\sum_{k=1}^{n-j}\sum_{i=0}^{j-k-1}\frac{\Psi_{j-i,k}(C)\nabla\omega_{j-i,k}(C)\cdot C_j'(t)}{E_k\left[\dfrac{s}{p_1}\right]}>0,$$

that is again positive by the definition of diversification and its higher-order effects. Finally, the diversification with the market price of contracts will make lenders indifferent because the lenders are indifferent between lending more or lending less by Lemma 1 and Theorem 1. Therefore, every agent is receiving payoffs better than or equal to the payoffs of the original equilibrium after the diversification with cash transfers. ∎

B.3. Counterparty Irrelevance

If there is no lender default loss—that is, $\Psi_{ij}(C)=0$ for any C and $i,j\in N$—then the payment equilibrium is unique because there will be no jumps in the aggregate wealth. Also, without a default loss, a change in counterparty connections does not matter as long as the total borrowing and lending amount remain the same. The following proposition states this property.

Proposition B4 (**Counterparty Irrelevance**). *If there is no lender default loss, then the payment equilibrium is unique for any given network. Furthermore, two networks (C,D) and $(\hat{C},\hat{D})$ with the same indegrees and outdegrees—that is, $\mathbb{1}(C\circ D)=\mathbb{1}(\hat{C}\circ\hat{D})$ and $(C\circ D)\mathbb{1}=(\hat{C}\circ\hat{D})\mathbb{1}$—will have the same payment equilibrium.*

Proof of Proposition B4. For a fair price, there exists a unique equilibrium price no matter what happens in shocks and bankruptcies. Now focus on liquidity constrained prices. When $\Psi_{ij}(C)=0$ for any $i,j\in N,\ C\geq 0$, equation (3) becomes

$$\sum_{j\in N}e_j^1=\sum_{j\in N}\min\left\{\epsilon_j,e_j^1+a_j^1p-\sum_{i\in N}c_{ij}\min\{p,d_{ij}\}+\sum_{k\in N}c_{jk}\min\{p,d_{jk}\}\right\},$$

and by intermediation order, the right-hand side is increasing in p. Also the right-hand side is bounded below by $\sum\limits_{j\in N}\min\{\epsilon_j,e_j^1\}$, when $p=0$. By intermediate value theorem, there exists a unique equilibrium price p between $[0,s]$ that satisfies the market clearing condition above.

For the second statement of the proposition, first note that the sum of non-negative nominal wealth with no lender default loss is

$$\sum_{j\in N}[m_j(p)]^+ = \sum_{j\in N} e_j^1 + \sum_{j\in N} a_j^1 p$$
$$- \sum_{j\in N} \min\left\{\epsilon_j, e_j^1 - \sum_{i\in N} c_{ij}\min\{p, d_{ij}\} + \sum_{k\in N} c_{jk}\min\{p, d_{jk}\}\right\},$$

which can be re-written as the sum of indegrees and outdegrees as below.

$$\sum_{j\in N}[m_j(p)]^+ = \sum_{j\in N} e_j^1 + Ap - \sum_{j\in N} \min\left\{\epsilon_j, e_j^1 - \sum_{i\in N} c_{ij}x_{ij} + \sum_{k\in N} c_{jk}x_{jk}\right\},$$

where $x_{ij} = \min\{p, d_{ij}\}$, and the equation will have the same value with a network with

$$\sum_{i\in N} c_{ij}x_{ij} = \sum_{i\in N} \hat{c}_{ij}\hat{x}_{ij}$$
$$\sum_{k\in N} c_{jk}x_{jk} = \sum_{k\in N} \hat{c}_{jk}\hat{x}_{jk},$$

so networks (C, D) and $(\hat{C}, \hat{D})$ have the same equilibrium price and final asset holdings. ■

This proposition shows the necessity of a lender default loss (or any counterparty risk) in order to generate meaningful interaction among agents. Because of the absence of a default loss, an agent's individual connection does not matter as long as the total borrowing and lending are the same. The result is not so surprising since the main reason for using collateral is to insulate the lender from the counterparty risk.

C. Details of the Numerical Exercises

C.1. Equilibrium Search Algorithm

The following algorithm shows how to solve the payment equilibrium in quantitative analysis under the maximum equilibrium selection rule.

0. Set $B^{(0)}(\epsilon) = \varnothing$. Start with step 1.
1. For any step k, given $B^{(k-1)}$, compute $p^{(k)}$ that satisfies equation (8).
2. For given $p^{(k)}$, compute $m_j(p^{(k)})$ with given $B^{(k-1)}$ and update $B^{(k)}$.
3. If $B^{(k-1)} = B^{(k)}$, then it is the maximum equilibrium. Otherwise, move to the next step $k+1$ and repeat procedures 1 and 2.

This algorithm, which is an extension of the algorithm of Eisenberg and Noe (2001), is guaranteed to find the maximum payment equilibrium price of the given network. Also, the algorithm finishes within n steps because the second-order bankruptcy (cascades) could only occur at the maximum of $n-1$ times.

C.2. Parameter Values

For the comparative statics in figure B1, I use $n = 10$ agents with the vector of beliefs on the asset payoff as $(s_1, s_2, \ldots, s_10) = (20, 19, 18, \ldots, 11)$. The baseline parameters are as follows. Each agent has the initial endowment of cash $e^0 = 5000$. The total supply of assets is $A = 5000$. The lender default loss function is

$$\Psi_{ij}(C) = \frac{c_{ij}}{\sum_{k\in N} c_{ik}} \left(\frac{\sum_{k\in N} c_{ik}}{A} \right)^2 .$$

The common liquidity shock distribution is a log-normal distribution with the mean of 6 and standard deviation of 5. I sample 5000 joint realizations from this distribution. The probability of receiving a liquidity shock is $\theta_i = 1$ for any agent $i \in N$.

Following Theorem 1, the contract matrix D is fixed as $d_{ij} = s_i$ for any $j < i \in N$ and 0 otherwise. For the comparative statics, I used the collateral matrix C of the single-chain network as the baseline collateral matrix. The baseline case is the collateral matrix with the maximum collateral exposure. Therefore, $c_{i,i-1} = 5000$ for any $1 < i \leq n$ and $c_{ij} = 0$ if $j \neq i-1$. Other matrices such as a multi-chain network show similar patterns.

For each comparative statics, each line represents the subjective expected price of each agent starting from agent 1 to agent 10. Each subjective expected price is computed by obtaining the simulated expectation over 5000 realizations with the respective s value for each given subjective belief. For example, the asset price can be up to 20 under agent 1's belief if there is no significant liquidity shock, but the asset price under agent 2's belief can only be up to 19 for the same liquidity shock realization.

For the change in collateral exposure, I fixed every parameter as the baseline case except for the collateral matrix C. I started with the reduced collateral exposure value such that $c_{i,i-1} = 2500$ for any $1 < i \leq n$ and $c_{ij} = 0$ if $j \neq i - 1$. The horizontal axis of the upper-left panel of figure B1 is the multiplier of the given collateral matrix. Thus, 2 is the case of the collateral matrix with the maximum collateral exposure.

For the change in mean of liquidity shocks ϵ, I fixed every parameter as the baseline case except for the mean of the log-normal distribution of the common liquidity shock G. The horizontal axis of the upper-right panel of figure B1 is the mean starting from 5 to 7.

For the change in probability of liquidity shocks θ, I fixed every parameter as the baseline case except for the probability θ of receiving a liquidity shock drawn from the common distribution G. The horizontal axis of the lower-left panel of figure B1 is the probability starting from 0 to 1.

For the change in cash holdings e^1 of each agent, I fixed every parameter as the baseline case except for the common cash holdings e^1 of each agent. The horizontal axis of the lower-right panel of figure B1 is the amount of cash holdings starting from 1000 to 10000.

For the change in the degree of diversification of agent 3, I fixed every parameter as the baseline case except for the collateral matrix C. First, I define the collateral matrix with full diversification of agent 3's collateral exposure as $\tilde{C}$. Under $\tilde{C}$, agent 3 is equally exposed to agents 3, 4, and so on. Further, I adjust the collateral matrix to satisfy the collateral constraint of each subsequent agent. The adjustment is done by scaling down each collateral exposure starting from agent 4 if the collateral outflow from an agent exceeds the collateral inflow to the agent. Then, I compute a convex combination of C and $\tilde{C}$ with the weight of $\tilde{C}$ as the degree of diversification. The horizontal axis of the top-right panel of figure B1 is this weight of $\tilde{C}$ for the convex combination of collateral matrices used in each simulation.

Online Appendix References

Eisenberg, L. and T. H. Noe (2001): "Systemic Risk in Financial Systems," *Management Science*, 47, 236–249.